Multiscale Finite element on 1D BVP

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Introduction

Accurate and efficient simulation of multiphase flow in large-scale heterogeneous natural formations is crucial for a wide range of applications, including hydrocarbon production optimisation, risk management of carbon capture and storage, water resource usage and geothermal power extractions. Unfortunately, considering the size of the domain along with the high resolution heterogeneity of the geological properties, such numerical simulations is often beyond the computational capacity of traditional reservoir simulators. Therefore, Multiscale Finite Element (MsFE) and Finite Volume (MsFV) methods and their extensions have been developed to resolve this challenge.

In this poster, some basic properties of the Multiscale methodology are described. The technique is employed to produce approximate solutions of 1D Boundary Value Problems (BVP) in different circumstances.

Model Problem

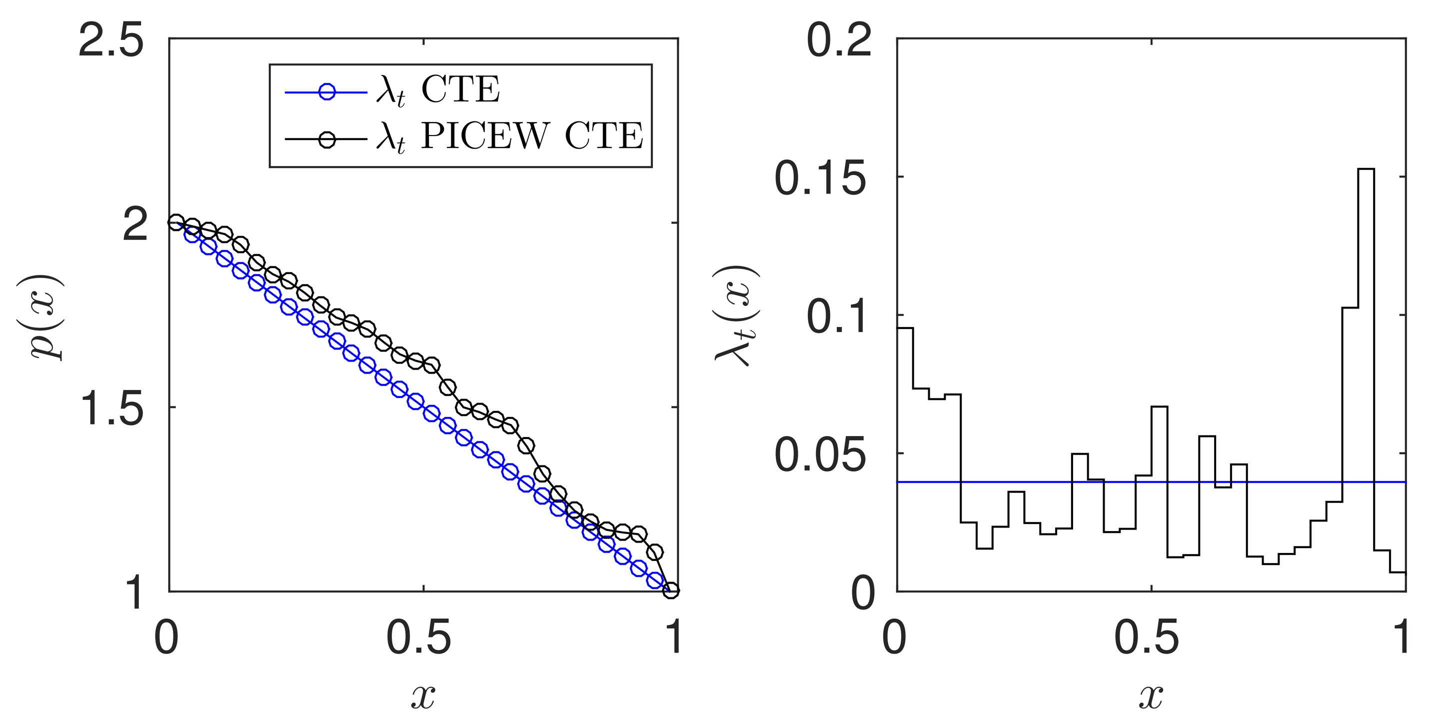
The pressure equation for incompressible one-dimensional flow is considered in a certain domain $\Omega = [a, b]$. Dirichlet Boundary Conditions (DBC) complete the 1DBVP:

$$-\nabla \cdot (\lambda_t \nabla p(x)) = q(x), \qquad x \in \Omega, \qquad p(a) = c, \qquad p(b) = d; \tag{1}$$

λ_t is the total mobility and $q(x)$ will be referred as source term. Defined a uniform grid of N elements, classical Finite Volume Method (FVM) can be used to produce a fine-scale solution by imposing conservation of the mass in each individual cell. The following first order scheme can be employed:

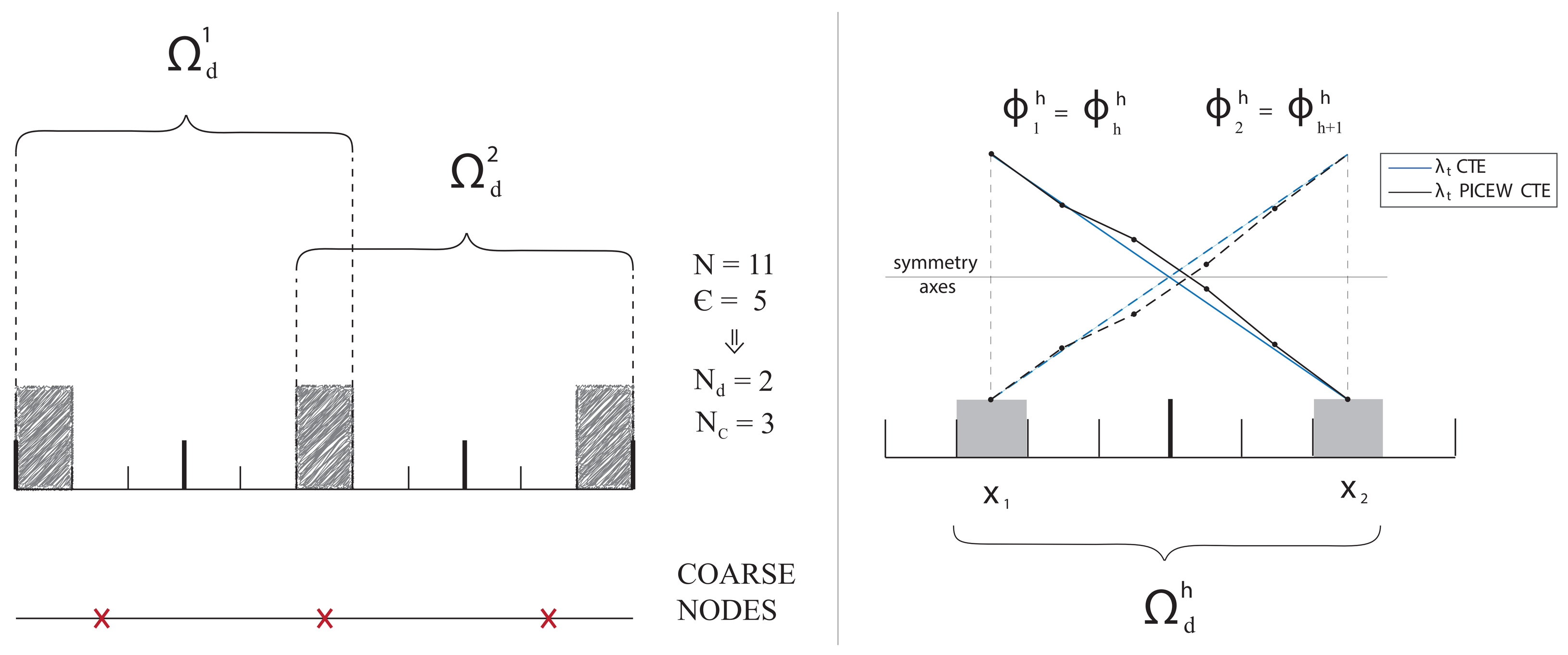
$$\frac{\lambda_t^{i-1/2}}{h}(p_i - p_{i-1}) + \frac{\lambda_t^{i+1/2}}{h}(p_i - p_{i+1}) = q_i \tag{2}$$

with λ_t defined at the grid interfaces and p, q in the nodal points for an algebraic system of N equations $Ap = q$ (DBC included). FVM solution of homogeneous problem ($q = 0$) is shown in Fig1 for two different configurations of λ_t. 1) Constant mobility (blue), 2) Picewise constant λ_t randomly generated (black).



Multi Scale Grid and Local Basis

MS grid consist on a fine, a coarse and a dual-coarse mesh. Given a fine scale discretization and the ratio ϵ (number of fine elements in a coarse cell), we can define a Multiscale grid with N fine, N_c coarse and N_d dual cells (Fig.2 left). Notice that dual cells overlap in one fine element, and each dual contains a piece of two different coarse cells. The nodes (center of the cell) of all three meshes are uniformly distributed (which is very convenient in order to use (2)). Different set ups for the MS grid are possible even supporting unstructured discretizations.



The basis functions are then locally computed on dual cells and each dual cell has 2 local basis ϕ_1^h and ϕ_2^h (Fig.2 right). They correspond to the solution of the homogeneous problem

$$-\nabla \cdot \left(\lambda_t \phi_k^h\right) = 0, \qquad \phi_k^h(x_m) = \delta_{km} \qquad k = \{1, 2\} \qquad h = 1, \ldots, N_d \tag{3}$$

where x_m are the boundary nodes. It can be proven that $\phi_1^h(x) + \phi_2^h(x) = 1$, we say they form a partition of the unity, providing that ϕ_1^h is symmetric to ϕ_2^h wrt $y = 0.5$ axis.

Global basis, Prolongation and Restriction

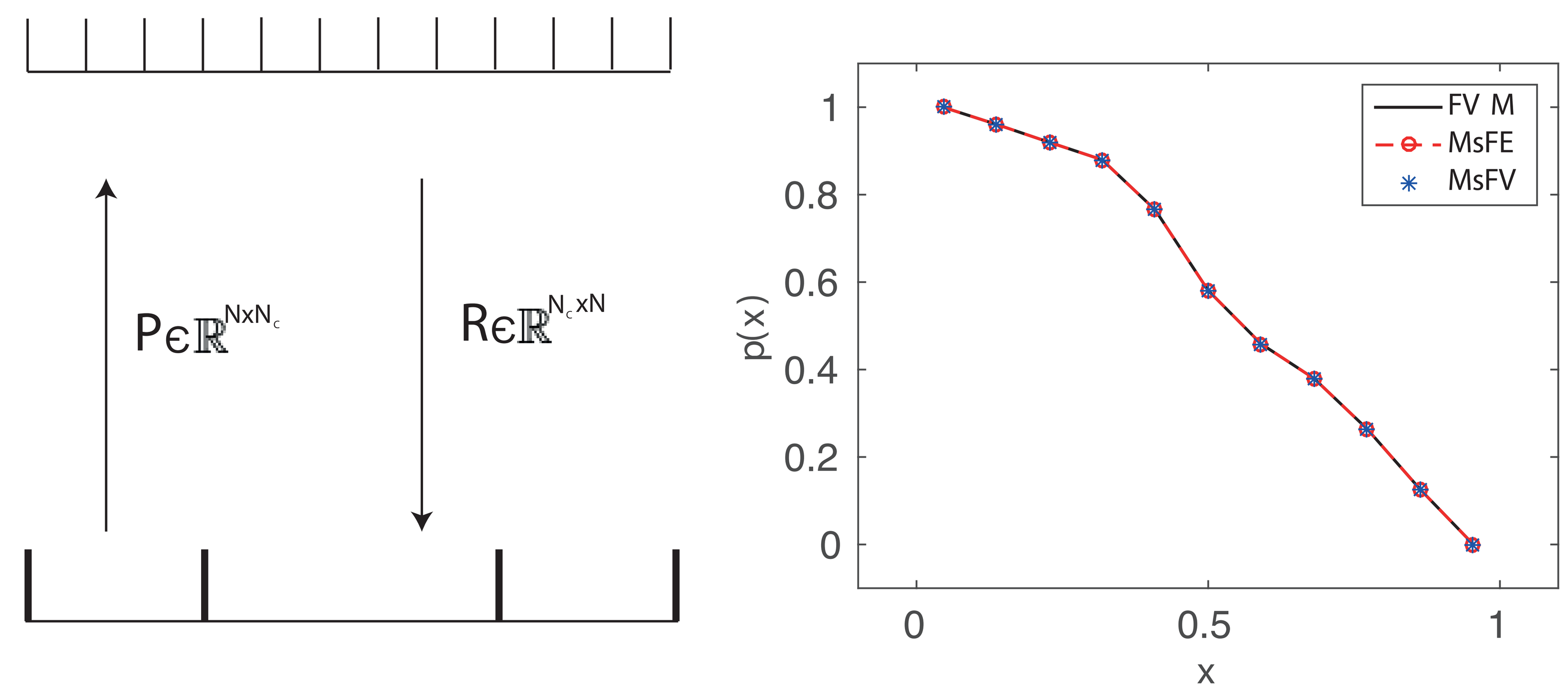
Global basis functions are the extension of local ones to the whole domain, they must be zero out of the corresponding dual cell and k subindex runs in the coarse elements $k = 1, \ldots, N_c$ such that ϕ_k^h is only non zero for $k = \{h, h+1\}$. The fine scale solution p can be interpolated in this basis and the coefficients $\hat{p}_k$ correspond to the pressure values at the coarse nodes.

$$p \approx \sum_{h=1}^{N_d} \sum_{k=1}^{N_c} \phi_k^h \hat{p}_k \tag{4}$$

We can rewrite this equation in matricial form $p = Pp_c$ where $P \in \mathbf{R}^{N \times N_c}$ is constructed by columns using the global basis. P is the prolongation operator that transforms a vector from coarse to fine resolution (Fig.3 left). The opposite operator R (restriction) can be defined in different ways for different Multiscale Methods. That choice determines the nature and properties of the MS Method: Finite Element approach (MsFE) considers $R = P^T$ and Finite Volume approach (MsFV) has R entries:

$$R_{ik} = \begin{cases} 1 & \text{if i} - \text{th fine element} \in \Omega_c^k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Defined these two matrices we can compute an approximation of the fine scale solution as $p \approx P(RAP)^{-1}Rq$. Since $RAP \in \mathbf{R}^{N_c \times N_c}$ we have reduced the effort of backslash operation compared with FVM solution $p = A^{-1}q$ (which is more notorious in higher order problems). Besides Multiscale methodology is highly parallelizable such that we can exploit modern many-core hardware an improve the efficiency of the simulation.



Left: Multiscale Operators. Right: approximate solutions of (1) using $q = 0$H.

MsFV preserves the mass of the solution while MsFE has been proven to be more efficient in large problems. For our 1DBVP given by (1) both approaches are exact in the Homogeneous case (Fig.3 right).

No-homogeneous case

When the source term is allowed to be no null, a correction term $\psi \in \mathbf{R}^N$ must be added to the multiscale approximations. That function can be computed locally in each dual cell and then assembled together for a vector in the fine grid. At Ω_d^h dual cell we solve the no-homogeneous problem using $c = d = 0$ in DBC and put those entries in the corresponding place of the final vector (i.e. ψ(EToVd(h, :))).

We first define a point-wise source term (i.e. only one non-zero entry located at a certain dual cell Ω_d^h) and then we generalized the formulation for all q, showing that both the MsFE and the MsFV can capture exactly the fine scale solution.

Fig.4 shows the results corresponding to point-wise source experiment. When the correction term is not included MsFE differs from FVM only in that precise dual cell being exact in the rest of them due to mass conservation (left). Including the correction term both multiscale solutions agree FVM (right).