Adaptive mesh refinement

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Introduction

Solving large systems using finite element method (FEM) can be computationally expensive. Some systems do not require the same complexity in all areas to achieve a good estimate and adaptive mesh refinement (AMR) may be beneficial in determining the optimal mesh for solving the FEM. We show how the use of AMR methods can be used to decrease CO_2 consumption when a small error is desired and also how the technique can be used for sparse representation of known functions.

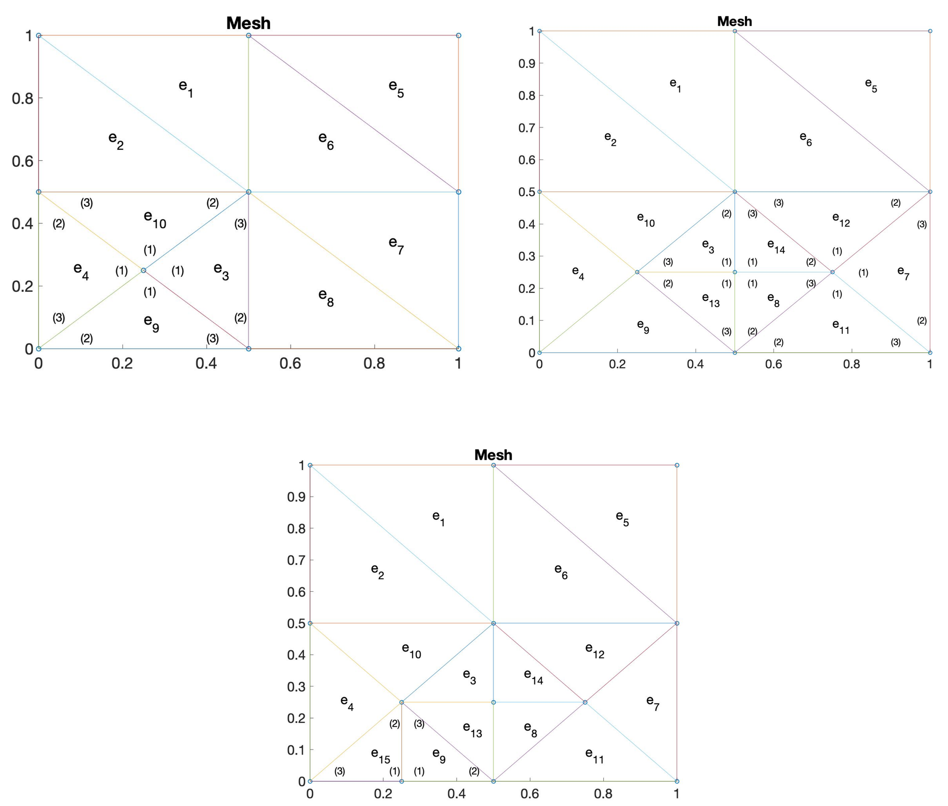
Methods

Algorithm

1. Initialize a coarse grid.
2. Estimate the error for all elements.
3. Refine elements with one of the following strategies:
   - Refine the elements with k largest errors.
   - Refine the elements with $error_i > \alpha \cdot tol$.
4. Estimate solution based on refined grid.
5. Repeat 2., 3. and 4. until all errors are smaller than the chosen tolerance.

Mesh refinement (newest node-bisection method)

3 cases:

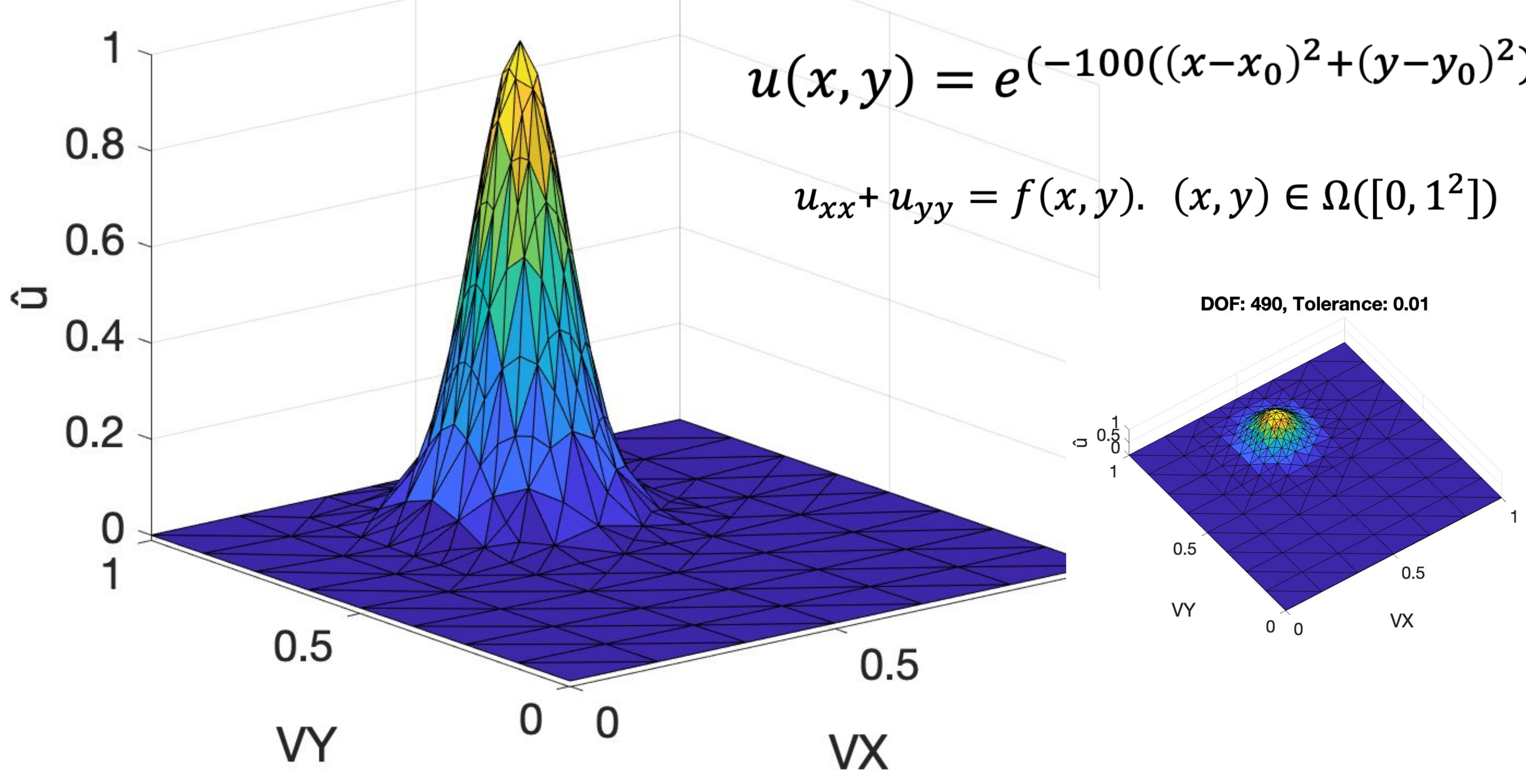


Results FEM + AMR

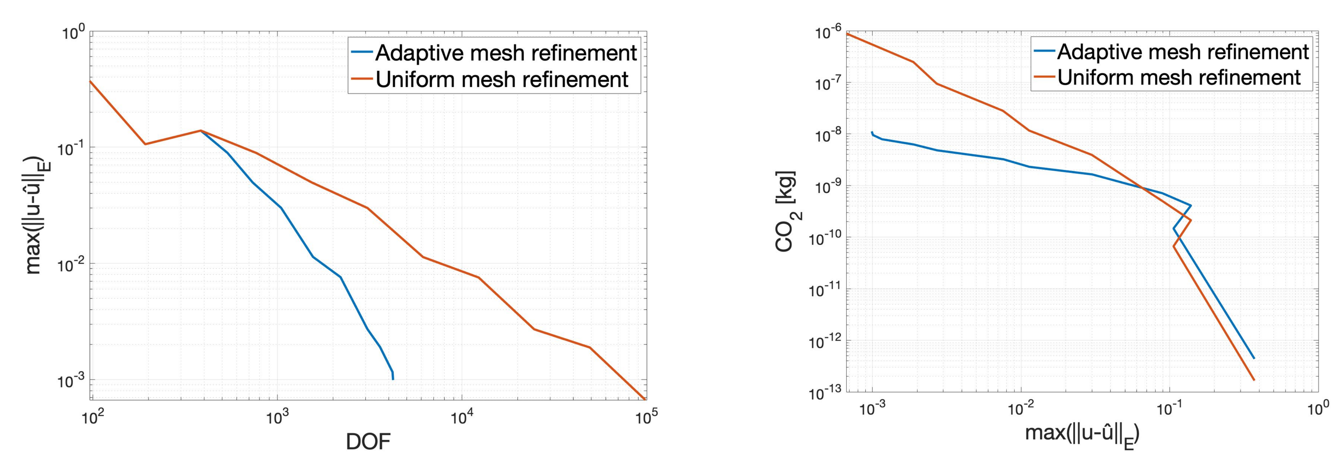
DOF: 490, Tolerance: 0.01

$$u(x,y) = e^{(-100((x-x_0)^2+(y-y_0)^2))}$$

$$u_{xx} + u_{yy} = f(x,y). \quad (x,y) \in \Omega([0,1^2])$$



Approximation of u(x,y) with FEM and AMR. Areas close and at the peak is more refined.

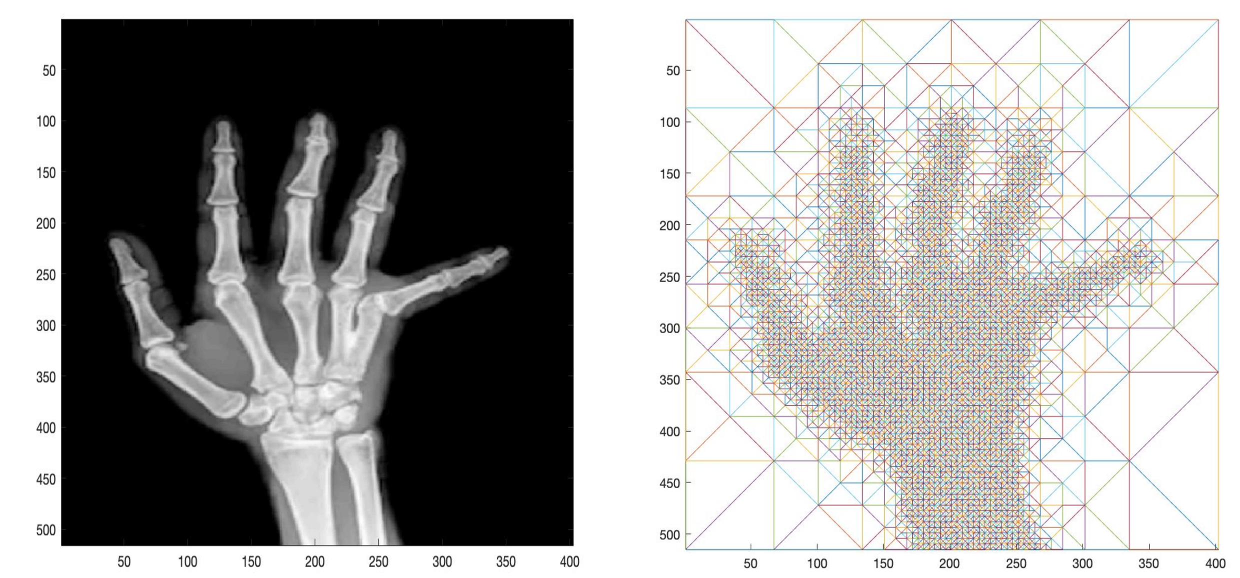


We see that at larger errors, the degrees of freedom is approximately the same for the adaptive and uniform refinement. Since the adaptive refinement also requires an estimate of the error, this means that the CO_2 consumption is larger for AMR at small errors.

Strategy		DOF	Iterations (time)
Maximum	1	490	147 (4.3 s)
	5	509	34 (1 s)
α·tol	1	758	9 (0.6 s)
	0.9	822	9 (0.6 s)

Different experiments has been carried out in order to test the DOF with different strategies as well as the run-time. When updating one element that has the maximum error with tolerance 1e-2 resulted in the smallest number of DOF with the largest number of iterations.

AMR for image compression



Simple way of estimate error:

The decrease in error is estimated by the change in the approximated volumen from the coarse mesh to the fine mesh, where A is the area of the element in the plane.

$$vol(\hat{u}\,(e_i)) \approx \frac{A}{3}(\hat{u}_1 + \hat{u}_2 + \hat{u}_3)$$

The error decrease is then the change in volume, where N is the number of elements the i'th element has been refined to:

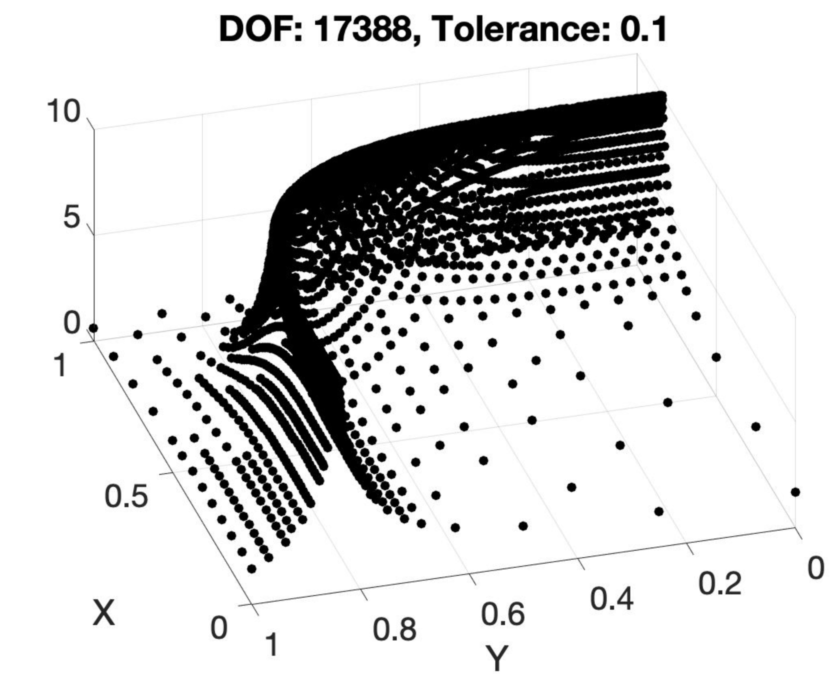
$$\Delta err_i = |vol(\,\hat{u}\,(e_i)) - \sum_j^N vol(\hat{u}\,(e_j))\,|$$

Interpolated image with 10.786 pixels compared to original image with approx. 200.000 pixels (5% of original image).

AMR for function estimation

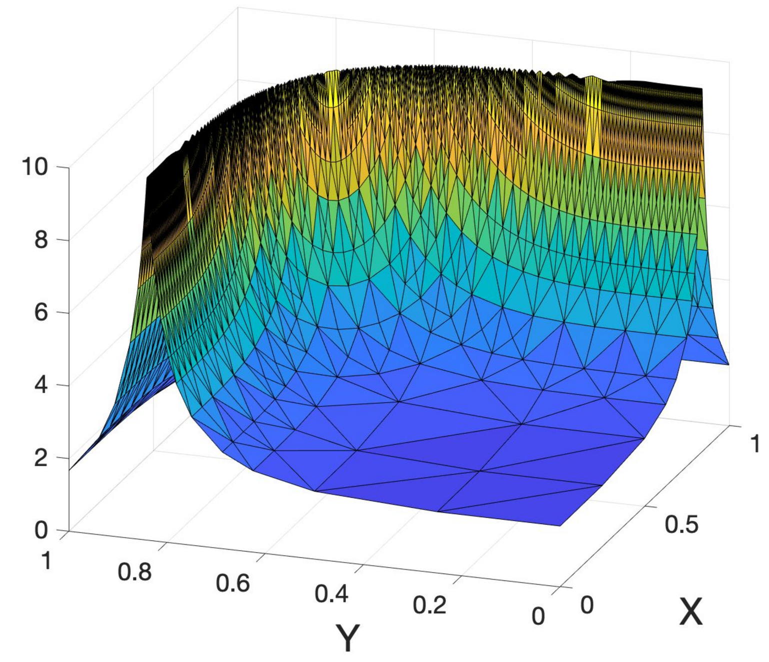
Estimate the following function by interpolating on an adaptive grid:

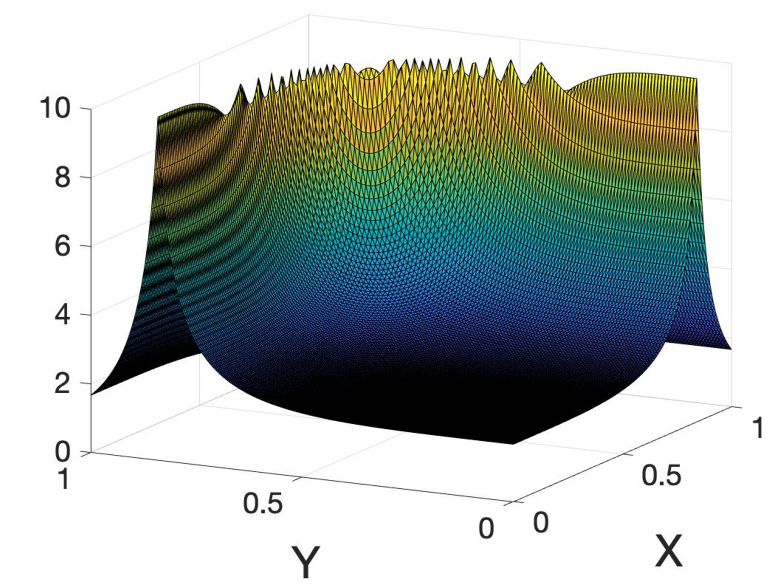
$$f(x,y) = (|0.5 - x^4 - y^4| + 0.1)^{-1}$$



Estimate error in element e_i with nodes (x_1,y_1), (x_2,y_2), (x_3,y_3):

$$err_i = \left|\frac{1}{3}\sum_{i=1}^{3} f(x_i,y_i) - f\left(\frac{1}{3}\sum_{i=1}^{3} x_i, \frac{1}{3}\sum_{i=1}^{3} y_i\right)\right|$$





Uniform mesh with 132x132 = 17.424 elements.