Preface

This thesis is written in partial fulfilment of the requirements for the Ph.D. degree in engineering from the Technical University of Denmark (DTU). The reported research has been carried out at the Department of Mathematical Modelling (IMM) at the same university in the period from September, 1995, through September, 1998. Associate professor Niels Kjølstad Poulsen was the academic advisor on the project.

Uffe Høgsbro Thygesen
Lyngby, September 30, 1998

Acknowledgements

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I also wish to thank students and staff at IMM for what they have taught me and for providing an inspiring and pleasant working environment. The list of people is too long to include here; let me just mention that I shared an office with Dr. Morten B. Lauritsen during the first half of my studies, which led to many interesting discussions on stochastic control theory.

I have spent much time with staff and students at the Department of Mathematics. Professor Jakob Stoustrup has influenced my view on control theory significantly, and has continued to do so after he left the department. I have also enjoyed many long discussions about mathematical control theory with Dr. Eric Beran, Dr. Marc Cromme, Dr. Mikael Larsen and Dr. Mike Lind Rank.

I am grateful to Professor D. Prätzel-Wolters and Dr. Jörg Hoffmann, with whom I had the pleasure to work during and immediately after my M.Sc. studies. I have had great benefit from the control theory and the mathematics which I learned at the University of Kaiserslautern.

I also wish to express my gratitude to Professor Robert Skelton, who convinced me about the importance of convex optimization in control theory. I spent a year in his group, which was at Purdue University at that time, where I worked with him on his Finite Signal-to-Noise Ratio models. These models also appear in this thesis and I gratefully acknowledge the inspiration from him. During this year I also learned much from Professor Martin Corless, in particular Lyapunov theory, and from Professor Mario Rotea, who introduced me to dissipation theory. I also enjoyed many stimulating discussions with my fellow students, in particular Dr. Goujun Shi and Dr. Jianbo Lu.

Last but not least, I thank Dr. Marc Cromme who carefully read through an early version of this thesis and whose comments improved the final result considerably.

Previous publication of the material

Parts of the material in this thesis have previously been published in references

[113] U.H. Thygesen and N.K. Poulsen. Min-max control of nonlinear systems with multi-dissipative perturbations. Tech. Rep. 23, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997. Pres. at the 6th Viennese WOCDGNLDAS, Vienna, 1997.

[114] U.H. Thygesen and N.K. Poulsen. On multi-dissipative perturbations in linear systems. Technical Report 1, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[115] U.H. Thygesen and N.K. Poulsen. Robustness of linear systems with multi-dissipative perturbations. In *Proceedings of The American Control Conference*, pages 3444–3445, 1997.

[116] U.H. Thygesen and N.K. Poulsen. Simultaneous $\mathcal{H}_\infty$ control of a finite number of plants. Technical Report 24, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[117] U.H. Thygesen and N.K. Poulsen. Simultaneous output feedback $\mathcal{H}_\infty$ control of p plants using switching. In *Proceedings of the European Control Conference*, 1997.

Other parts are included in papers which are in the process of being reviewed for publication:

[109] U. H. Thygesen. On dissipation in stochastic systems. Under review for publication in a journal. A short version is submitted to a conference, 1998.

[111] U.H. Thygesen. On multi-dissipative dynamic systems. Submitted to a journal. A short version is submitted to a conference, 1998.

Finally, some results are included in manuscripts in preparation:

[112] U.H. Thygesen. On the conditional expectation of first passage times. Manuscript in preparation, 1998.

The material published in the references [72, 108, 110] is not included in this thesis.

Summary

The topic of the present dissertation is robustness and performance issues in nonlinear control systems.

The control systems in our study are described by nominal models consisting of nonlinear deterministic or stochastic differential equations in a Euclidean state space. The nominal models are subject to perturbations which are completely unknown dynamic systems, except that they are known to possess certain properties of dissipation. A dissipation property restricts the dynamic behaviour of the perturbation to conform with a bounded resource; for instance energy. The main contribution of the dissertation is a number of sufficient conditions for robust performance of such systems.

Since the perturbations in these uncertain models possess several dissipation properties simultaneously, we study fundamental properties of such multi-dissipative systems. These properties are related to convexity and topology on the space of supply rates. For instance, we give conditions under which the available storage is a continuous convex function of the supply rate.

Dissipation theory in the existing literature applies only to deterministic systems. This is unfortunate since robust control applications typically also contain uncertainty which is better modelled in a probabilistic framework, such as measurement noise. This motivates an extension of the theory of dissipative dynamic systems to stochastic systems. This dissertation presents such an extension: We propose a definition and generalize fundamental results from deterministic dissipation theory to stochastic systems.

Furthermore, we argue that stochastic dissipation is a natural fundament for a theory of robust performance of stochastic systems. To this end, we present a number of performance requirements to stochastic systems which can be formulated in terms of dissipation, after which we give sufficient conditions for these requirements to be robust towards multi-dissipative perturbations.

A final contribution of the dissertation concerns the problem of simultaneous $\mathcal{H}_\infty$ control of a finite number of linear time invariant plants. This problem is a prototype of robust adaptive control problems. We show that the optimal (minimax) controller for this problem is finite dimensional but not based on certainty equivalence, and we discuss the heuristic certainty equivalence controller.

Resumé (in Danish)

Emnet for denne afhandling er robusthed og ydelse (performance) af ikke-lineære reguleringssystemer.

Reguleringssystemerne er beskrevet af nominelle modeller bestående af ikke-lineære deterministiske eller stokastiske differentialligninger i et euklidisk tilstandsrum. Disse nominelle modeller underkastes perturbationer som er ukendte dynamiske systemer om hvilke det dog vides at de besidder visse dissipationsegenskaber. En dissipationsegenskab indskrænker perturbationens dynamiske opførsel ved at påtrykke en begrænset ressource, for eksempel energi. Hovedbidraget i denne afhandling er et antal tilstrækkelige betingelser for robust ydelse af sådanne systemer.

Eftersom perturbationerne i disse usikre modeller besidder flere dissipationsegenskaber samtidigt, studerer vi fundamentale egenskaber af sådanne multi-dissipative systemer. Disse egenskaber omhandler konveksitet og topologi på rummet af tilførselsrater (supply rates). For eksempel opstiller vi betingelser under hvilke det tilgængelige lager (available storage) er en kontinuert konveks funktion af tilførselsraten.

Den eksisterende litteratur beskriver kun dissipationsteori for deterministiske systemer. Det er uheldigt fordi anvendelser af robust regulering typisk også indeholder usikkerhed som bedst modelleres sandsynlighedsteoretisk, såsom målestøj. Det er motivationen for at denne afhandling udvider dissipationsteorien til stokastiske systemer: Vi foreslår en definition og generaliserer nogle af de grundliggende resultater fra deterministisk dissipationsteori til stokastiske systemer.

Derefter argumenterer vi for at stokastisk dissipation er et naturligt udgangspunkt for en teori for robust ydelse af stokastiske systemer. Til dette formål opstiller vi et antal kvalitetskriterier for stokastiske systemer som kan formuleres som dissipationsegenskaber, og dernæst angiver vi tilstrækkelige betingelser for at disse kriterier er robuste overfor multi-dissipative perturbationer.

Herudover behandler denne afhandling også problemet om simultan $\mathcal{H}_\infty$ regulering af et endeligt antal lineære tidsinvariante anlæg. Dette problem fungerer som en prototype på robust adaptiv regulering. Vi viser at den optimale regulator (d.v.s. minimax-regulatoren) for dette problem er endelig-dimensional men ikke bygger på certainty equivalence. Derudover diskuterer vi heuristisk certainty equivalence regulering.

Contents

Chapter 1

Introduction

The subject of this dissertation lies within the field of mathematical control theory.

In this chapter we give a broad introduction to the field of mathematical control theory. Those among the readers who are more interested in the specific contributions of the dissertation may prefer to jump to section 1.5, which outlines the dissertation, and from there to the succeeding chapters which presents the new material.

1.1 What is control theory?

The subject of control theory is the interconnection of the dynamic systems Σ and K in figure 1.1. Here Σ is a given dynamic plant (a mathematical model of a physical system) and K is the controller (which is also a mathematical model of a physical system). The objective is to design the controller K, i.e. to find a suitable K, such that the interconnection has some desirable properties. These properties typically describe how the interconnection responds to an exogenous input w and are quantified through the output z. The controller affects the response by choosing the control signal u. The controller has at least partial access to information about the state of the plant, quantified by the measurement signal y.

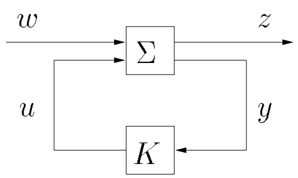

Figure 1.1: A control problem

The questions addressed by control theory are *analysis* questions and *synthesis* questions. Analysis questions investigate properties of the interconnection for a given controller K, while the synthesis question is how to choose the controller K such that the interconnection has certain properties. The motivation for considering analysis questions is twofold: First, analysis questions are most often much easier to answer than synthesis questions, but good answers to analysis questions often enable the control theorist to find answers to the corresponding synthesis questions. Second, the control engineer may have found a candidate controller by solving one specific synthesis question and then wish to know if this controller provides satisfactory answers to other analysis questions.

In order to answer analysis as well as synthesis questions, control theory employs several disciplines from the field of applied mathematics. Qualitative and quantitative theory for deterministic and stochastic dynamic systems is essential as is optimization theory. In addition, statistical inference or deterministic estimation theory is necessary to address problems where the measurements y contain only incomplete information about the state of the plant Σ.

The resulting theory depends greatly on the specifics of the interconnection in figure 1.1: If the systems are linear or nonlinear, deterministic or stochastic, and if the dynamic systems are described in continuous or discrete time. The next section describes these differences in some further detail, as well as clarifies and motivates how the present dissertation is placed in this discourse.

1.2 Paradigm and state of the art

Robustness or performance?

While most control theorists and engineers agree that robustness and performance are desirable properties of a controlled system, and that as a result analysis and synthesis must address these issues, there is much less consensus regarding the exact meaning of these properties and their relation.

Performance measures the quality of the controlled system: How fast, how accurate or how effective is the system. In this work we use the term performance to describe how a cost, accumulated during the operation of the system, depends on the initial state of the system, or on exogenous deterministic or stochastic disturbances. The lower cost, the better performance.

The issue of *robustness* arises because the mathematical model, which is the object of the mathematical analysis, never fully describes the physical control system. Loosely, robustness means that the mathematical analysis predicts the behaviour of the physical system with sufficient accuracy. We assign a much more precise meaning to the word robustness: We model the physical system by a *family* of mathematical models (typically obtained as an interconnection of a nominal model and an unknown perturbation), and say that a property is robust if it holds for any model in this family.

An often heard statement is that one must trade off robustness and performance: For instance, if one wishes a fast response of a servo system, one must accept that the system is sensitive to parasitic dynamics. We do not disagree that such trade-off considerations between sensitivity and nominal performance are helpful. However, we prefer to discuss the issue: How fast a response can we obtain in presence of parasitic dynamics? Thus the objective is to guarantee a level of performance which is robust towards a given family of perturbations. In summary, the question *Robustness or performance?* should be answered: *Robust performance!*

Linear or nonlinear theory?

A seemingly never-ending controversy among control theorists concerns linear versus nonlinear theory. Advocates of nonlinear theory emphasize that nonlinear models provide more accurate descriptions of technical systems which makes it plausible that better control systems can be obtained with nonlinear theory. On the other hand, nonlinear theory quickly becomes so involved that the designer can be forced to oversimplify the problem, for instance by neglecting certain dynamics, and in these situations it is plausible that linear theory is more effective. Also, in industrial applications it cannot be neglected that nonlinear control theoretic investigations consume great resources which perhaps would be more beneficial if allocated to complete different parts of the design project. A fact that keeps the controversy going is that some fields of applications manage quite well with linear models whereas nonlinearities are essential to the problems of other fields.

Tools for analysis and design of controllers for linear plants are well developed and implemented in commercial software packages such as `MATLAB`. The engineering appeal of frequency domain techniques is an important factor, as is the fact that it is possible to give standardized recipes which work for most linear problems. There is an abundance of methodologies, ranging from parameter tuning in PID-controllers to μ-synthesis [128]. For engineers who wish to pose their own non-standard design criteria, the framework of linear matrix inequalities is an option [20, 19]. Important open problems within the linear paradigm, which are topics of current research, concern mixed and multi-criteria problems, the problem of designing controllers of fixed structure, and interdisciplinary topics such as simultaneous design of system and controller.

Regarding nonlinear control theory significant progress has been made but a fully operational general theory is still far away; indeed it is plausible that such a theory is utopian. The field of Lyapunov stability [74, 59] illustrates the hurdle: The theory is fairly complete from an analytical point of view, but the problem of computing Lyapunov functions for a general system is overwhelming. The same discussion applies to optimal control and differential games where it is known as Bellman's *curse of dimensionality*: The computational complexity grows exponentially with the dimension of the underlying state space. Despite increased computational power and improved numerical methods [65, 11] we cannot expect to be able solve all control problems by direct solution of partial differential equations on state space: It is not unusual for technical control problems to have 75 states as in [18]. One may imagine the effort required to compute, implement and understand a nonlinear controller feeding back a state of this dimension.

As a consequence a myriad of special cases have been investigated and

sometimes the special structure enables progress. For instance, within the last decades the differential geometric framework [51] has evolved. The associated tools such as feed-back linearization are valuable, although they are prone to robustness problems and require special structure. Backstepping [63] and other recursive design techniques provide a methodology for systematic design, but requires considerable computational effort and a certain skill of the designer. Inverse optimality [36] is another promising concept; with this approach one solves the linearized problem at first and then constructs a nonlinear control law such that certain robustness properties of the linearized system hold globally for the nonlinear system.

With this state of the art, researchers and engineers must in each project choose pragmatically between the linear and the nonlinear paradigm. There is little doubt that nonlinear theory is becoming increasingly important as models grow in fidelity and complexity, as desired operating regions grow larger, and as better controller hardware allows more complex controller algorithms to be implemented. What is more, many concepts and ideas are clearer for nonlinear systems than for linear systems where matrix manipulations tend to obscure the picture; this is perhaps most evident in the field of stability and of optimal and $\mathcal{H}_\infty$ control. Therefore, our ambition in this dissertation is to develop control theory which is based on principles applicable to general nonlinear systems.

Time domain or frequency domain?

Within the linear paradigm, a great strength of control theory is the ability to combine considerations in frequency domain and time domain. Unfortunately, frequency domain tools are less than effective in a general nonlinear context where even the elementary concept of bandwidth is problematic. It remains a formidable project to find suitable substitutes.

Therefore, this dissertation considers systems in time domain exclusively. Without making a virtue out of necessity, an advantage of time domain techniques is that they appeal to that intuition for dynamic systems which engineers develop by studying physical systems. Not only does this facilitate the study and teaching of control theory, but it is also advantageous in industrial environments where a sharp distinction between controller and plant cannot be maintained, and where the control engineer works in an interdisciplinary team. A splendid example where experience from physics is of great value in control theory is the field of Lyapunov stability [74], calculus of variations and dynamic optimization [41, 3], which originately concerned mechanical and in particular astronomical systems, but which in the last decades have been developed by control theorists [12, 21, 122].

Deterministic or stochastic representation of uncertainty?

The explicit consideration of uncertainty lies at the core of control theory. Uncertainty may be represented by unknown parameters, unknown inputs or uncertain dynamical elements, and although much recent work [26, 60, 68, 69, 86, 98, 106, 129, 132] is devoted to mixed problems combining two or more types of uncertainty, there does not yet exist a general operational framework within which all these representations of uncertainty can be embedded.

Models of uncertainty can be divided into two groups: The stochastic ones, i.e. those that build on an underlying probability space, and the deterministic ones, which typically result in worst-case considerations. It is not uncommon for control theorists and engineers to have a very firm preference for one of the two groups, and occasionally this results in attempts to demonstrate that the one group can cover all models of uncertainty.

This dissertation is based on the pragmatic point of view that control theory should, to the widest extent possible, allow for both groups of uncertainty. With such a theory at hand, the control engineer can in each application choose to use deterministic or stochastic models, or both. This becomes increasingly important as control objects grow in complexity, since a complex control problem may contain both elements which require stochastic descriptions and elements which are suited for deterministic worst-case considerations.

1.3 Two recent advances in control theory

In this section we outline two recent developments in the field of control theory which have, too, provided background for the present work: Nonlinear $\mathcal{H}_\infty$ control theory and semidefinite programming. In short, nonlinear $\mathcal{H}_\infty$ theory is an analytical framework for addressing issues of robustness of nonlinear systems towards dynamic uncertainty. Semidefinite programming is a special case of convex optimization which can be used as a computational tool in control problems.

1.3.1 Nonlinear $\mathcal{H}_\infty$ control

One of the important products of control research of the 1980's was the formulation and solution of the linear $\mathcal{H}_\infty$ control problem. The background for this work was the robustness of LQG (Linear dynamics, Quadratic cost functions, Gaussian noise distributions, [66, 2]) controllers - robustness is here in the sense of the classical gain and phase margins. It was known that linear quadratic *state feedback* regulators provide universal robustness gain margins of $(\frac{1}{2}, \infty)$ and phase margins of ± 60 degrees [1, 93]; an impressive result which can be generalized in several directions, [118, 120]. This led to the question if similar universal margins existed for LQG controllers where the state is not available for feedback. Unfortunately, this is not the case [27]; optimal controllers are not necessarily robust. This motivated the formulation [127] of the $\mathcal{H}_\infty$ control problem. Here the design objective is to guarantee stability in presence of perturbations with $\mathcal{H}_\infty$ norm less than a specified number; this definition of robustness implies gain and phase margins, but is more general and more appealing from a mathematical point of view. A later generalization was the μ superstructure which allows several uncertain elements at different places in the closed-loop system; see [128] and the references therein.

Although the $\mathcal{H}_\infty$ framework originated in the frequency domain,[1] the celebrated DGKF solution [29] exploited the fact that the $\mathcal{H}_\infty$ norm of a transfer function is also the $\mathcal{L}_2$-gain of the associated input/output operator and thus relied on time-domain techniques, in particular completion of the square under an integral. This solution hinted towards two-player zero-sum differential games, thus suggesting that it would be possible to extent the $\mathcal{H}_\infty$ problem to nonlinear systems. An early development in this direction was [6]; the textbook [9] contains a large number of such results, mainly as extensions to the linear theory and focusing on different patterns of information available to the players.

Further insight into the nonlinear $\mathcal{H}_\infty$ problem was achieved in [119, 120] by stressing the connection to the theory of dissipative systems [124] and to that of Hamiltonian dynamics [3]. This also brought the field in touch with that of passive systems which has played a central rôle in modern control theory; bounded $\mathcal{L}_2$-gain and passivity constitute by now the most carefully investigated dissipation properties.

[1] The symbol $\mathcal{H}_\infty$ refers to the Hardy space [50] of transfer functions $G : \mathbb{C} \to \mathbb{C}^{m\times n}$ which are analytical in the right half plane, equipped with the supremum norm $\|G\|_\infty = \sup_{\omega\in R} \bar{\sigma} G(i\omega)$.

At the time of writing, there exists a well-established solution to the state feed-back nonlinear $\mathcal{H}_\infty$ control problem in terms of a Hamilton-Jacobi-Isaacs partial differential equation or inequality, which results from applying dynamic programming to the differential game. Nevertheless, issues related to the smoothness and properness of the value functions have yet to be worked out; here the notion of viscosity solutions [23] to partial differential equations has proved effective [7, 105]. Also, the numerical burden of actually computing bounds on value functions is still prohibitive except for problems with very low-dimensional state spaces; up to four, say, depending on the system at hand. Thus Bellman's *curse of dimensionality* also applies to these problems. As the paradigm of robust control includes a use of high-order dynamic weights we conclude that there is a need for heuristics and sub-optimal strategies which can deal with higher-dimensional problems.

Another remaining obstacle for the practical application of nonlinear $\mathcal{H}_\infty$ control theory is the design of state estimators in the situation where the state is not directly measurable, rather the controller is a causal map from a measured signal y to the control signal u. While static or finite-dimensional controllers may be optimal in special situations, [120, 121], it does not in general suffice to make use of a state observer of the same dimension as the control object [120]. In fact, general output feedback problems are very difficult and not fully resolved; not with respect to theoretical analysis and certainly not with respect to practical implementations. The most general framework for approaching these problems is that of the *information state*, see [55] and, in the context of stochastic optimal control problems on finite state spaces, [16]. With this technique the output feedback problem is first reduced to a state feedback problem. The state in this reduced problem is the information state which is a real-valued function on state space (termed the cost-to-go function by other authors, e.g. [25, 120, 9]) and hence the new problem requires infinite-dimensional dynamic programming.

In some situations the information state can be restricted to a finite-dimensional function space which facilitates the problem, see e.g. [56] or chapter 4 in this dissertation. In other situations one can *a priori* guarantee that a certainty equivalence principle holds [54, 14]. This reduces the complexity of the solution so that only two partial differential equations must be solved; one off-line (which governs the original problem with full state information) and one on-line (which governs the state estimation problem). While certainty equivalence principles thus simplifies control problems, it can be argued that certainty equivalence architectures lack the most intriguing feature of control with incomplete information: That additional information can be obtained by proper use of the control signal. This is the effect of probing, or *duality*.

1.3.2 Semidefinite programming and LMIs

A framework which has attracted much attention among control researchers relies on numerical solution of a special type of convex optimization problems, namely *semidefinite programs*. These optimization problems consist of optimizing a linear functional

$$\inf_x c'x$$

over all $x \in \mathbb{R}^n$ which satisfy a *linear matrix inequality* (LMI) constraint

$$A(x) \leq 0, \quad (\text{or} \quad A(x) < 0 \ )$$

where $c \in \mathbb{R}^n$ is a fixed co-vector and $A : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is an affine function taking symmetric matrix values. Such semidefinite programs are convex[2] and it is feasible to solve them numerically; powerful polynomial-time algorithms based on interior-point methods exist [82]. See also [13] for further references.

The surprising fact is that a large number of performance requirements in linear control theory can be formulated as linear matrix inequalities, see [19]. Thus semidefinite programming can be used to solve especially analysis problems but also some of the classic design problems, notably $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis. The simplest example is the well-known stability result [74] that, given a real matrix $A \in \mathbb{R}^{n\times n}$, the existence of a real symmetric *Lyapunov* matrix $P = P'$ such that

$$\begin{bmatrix} -P & 0 \\ 0 & PA + A'P \end{bmatrix} < 0$$

is necessary and sufficient for A to have all eigenvalues in the open left half of the complex plane. In chapters 3 and 6 we demonstrate that linear matrix inequalities provide the natural tool to deal with robustness analysis in linear systems where uncertainty is represented by *multi-dissipative perturbations* and *finite signal-to-noise ratios.*

[2]Meaning, we minimize a convex functional over a convex set.

The limitation of the LMI approaches to control *design* is that for the majority of controller design problems it is difficult or impossible to make the problem convex by variable substitutions. Rather, the resulting problems are bi-linear matrix inequalities which in general are non-convex. Numerical solution of bi-linear matrix inequalities is a topic of current research; see [13] and the references therein.

Finite-dimensional convex optimization, and in particular linear matrix inequalities, can also be employed as a computational tool for nonlinear control problems. Consider as an example the problem of finding a non-negative function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies the Hamilton-Jacobi inequality on $\mathbb{R}^n$

$$\sup_{w \in \mathbb{R}^m} \frac{\partial V}{\partial x}(x) f(x) + \frac{\partial V}{\partial x}(x) g(x) w - |w|^2 + |h(x)|^2 \leq 0 \quad .$$

The existence of such a *storage function* V implies (and is under certain conditions equivalent to) that the system

$$\dot{x}(t) = f(x(t)) + g(x(t)) w \ , \quad z(t) = h(x(t))$$

has $\mathcal{L}_2$-gain less than or equal to 1. Here $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n \times m}$. The set of those functions V which satisfies this inequality is convex; in fact the inequality is equivalent to

$$\begin{bmatrix} \frac{\partial V}{\partial x} f + |h|^2 & \frac{1}{2} \frac{\partial V}{\partial x} g \\ \frac{1}{2}(\frac{\partial V}{\partial x} g)' & -I \end{bmatrix} \leq 0 \quad . \tag{1.1}$$

A computational strategy for this infinite-dimensional convex feasibility problem is to search for a V of the form

$$V(x) = \sum_{i=1}^{N} \alpha_i V_i(x)$$

where V_i are basis functions, and require that the inequality (1.1) and $V \geq 0$ holds only at a finite set of points x_j, $j = 1, \ldots, M$. Inserting $V = \sum_i \alpha_i V_i$ in (1.1) and evaluating at x_j leads to M linear matrix inequalities in α_i

$$\left. \begin{bmatrix} \sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} f + |h|^2 & \frac{1}{2} \sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} g \\ \frac{1}{2}(\sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} g)' & -I \end{bmatrix} \right|_{x=x_j} \leq 0 \ , \quad j = 1, \ldots, M$$

which must holds together with with the M constraints

$$\sum_{i=1}^{N} \alpha_i V_i(x_j) \geq 0 \quad .$$

Thus LMI solvers such as [38, 32] may be used to search for storage functions V, and hence to compute the $\mathcal{L}_2$ gain of nonlinear systems. It can be argued that other numerical methods based on partial differential *equations* would be at least as effective for this particular problem of $\mathcal{L}_2$ gain analysis, but this dissertation contains numerous examples of nonlinear analysis problems which can be solved by convex optimization but not with equations. Admittedly, realistic control problems quickly lead to so large problems that the existing numerical tools for semidefinite programs will be ineffective, but as these tools are improving rapidly we expect that the approach may have practical applicability in the not so far future.

1.4 Problem formulation

Consider the control problem depicted in figure 1.2. The problem is to find a controller K in some set which maps measurements y to control signals u such as to achieve some design specifications on the output z. $\Sigma(\theta)$ is a plant which may be nonlinear and stochastic. The exogenous input w, the parameters θ, the dynamic perturbation Δ and the static nonlinear function ϕ represent uncertainty. All these uncertain elements are unknown but known to belong to some specified set. Additional uncertainty may be introduced by stochastic disturbances internal to $\Sigma(\theta)$.

With the current state of the art, this control problem is much too ambitious. The far more modest objective of this dissertation is simply to develop a framework within which this control problem can be formulated. Furthermore, to approach various subproblems, for instance by excluding some of the uncertain elements and considering analysis problems rather than synthesis problems.

As a starting point the theory of dissipative systems (in the sense of Willems [124] and Hill and Moylan, e.g. [46]) was adopted. See section 2.2 on page 28 below for an introduction and further references to dissipation theory. This is a natural choice in that problems of robust performance are easily formulated in terms of dissipation. Furthermore, dissipation theory

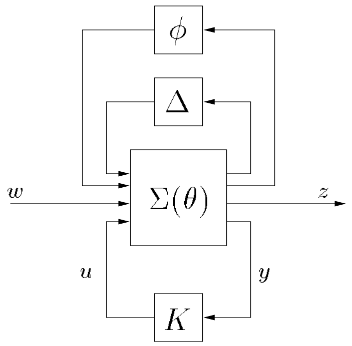

Figure 1.2: The ultimate control problem

assisted major theoretical achievements reached during the first half of the nineties in the field of nonlinear $\mathcal{H}_\infty$ control, which can indeed be seen as one of the subproblems mentioned above (see section 1.3.1 above).

Our research concentrated on three subproblems:

1. The problem arising by only considering a dynamic perturbation Δ which is known to possess several dissipation properties.
2. The problem arising when only considering uncertain parameters and robustness in the $\mathcal{H}_\infty$ sense.
3. The problem of incorporating stochastic noise signals in a dissipation-based framework for robustness.

The first item led to the study of multi-dissipative dynamic systems, see chapter 2, and to the study of robustness towards multi-dissipative perturbations, see chapter 3. The second item is that of adaptive $\mathcal{H}_\infty$ control or robust adaptive control; a topic which has been researched intensively over the past few years by several groups. Chapter 4 presents some new contributions to this topic, especially regarding the rôle of certainty equivalence, which were obtained by making the further simplification that the parameter θ belongs to a known, finite set. Finally, the last item motivated the notion of dissipative stochastic systems, a class of systems which is defined and investigated in chapter 5, and the investigation of a class of robust

stochastic control problems, namely those which involve multi-dissipative perturbations, see chapter 6.

1.5 Outline of the dissertation

The dissertation is divided into two parts. The first concerns *deterministic models.* This means that the nominal models are deterministic, i.e. ordinary, differential equations, and that uncertainty is represented by perturbations which belong to a specified set. The primary novelty of this part is the study of systems, in particular perturbations, which are dissipative with respect to *several* supply rates. Although dissipative systems are well-studied objects [124, 53, 122], multi-dissipative systems have not been discussed previously.

In part I, chapter 2 on *multi-dissipative dynamic systems* is devoted to fundamental properties of these systems; these properties concern convexity and continuity associated with the supply rates. Chapter 3 on *robustness towards multi-dissipative perturbations* develops *sufficient* conditions for robust stability and performance of systems subject to such perturbations. The conditions involve certain weights, or multipliers, associated with the dissipation properties and the results are shown using state-space time-domain techniques in the tradition of Lyapunov [74, 59]. The results have some conservativeness inherent which is illustrated by a simple example where non-conservative conditions can be obtained using an input-output approach. Chapter 4 on *simultaneous $\mathcal{H}_\infty$ control* assumes that the plant to be controlled is unknown, but belongs to a given finite collection. This a prototype of an adaptive $\mathcal{H}_\infty$ control problem and contains the problem of *duality* which remains a hurdle in stochastic adaptive control. We obtain an implicit solutions in terms of a partial differential equation, and discuss the structure of its solution as well as heuristic certainty equivalence control.

Part II concerns *stochastic models* where the nominal systems are described by stochastic differential equations in the sense of Itô. The aim of this part is to develop tools for problems which include both stochastic and deterministic representations of uncertainty. To this end, we develop in chapter 5 a theory of *dissipation in stochastic systems*, generalizing the framework of Willems [124]. We show that dissipative stochastic systems are as well-behaved as their deterministic counterparts; for instance dissipation has implications for stability and is preserved under interconnections of systems. This is exploited in chapter 6 on *robustness of stochastic systems*, where performance as well as uncertainty is described in terms of stochastic dissipation properties. Examples include $\mathcal{H}_2$ performance as well as disturbances with finite signal-to-noise ratios in the sense of Skelton [103]. The perspective of this framework is that it allows a modular approach to robustness analysis, using convex optimization as a numerical tool.

Concluding remarks and suggestions for future work are given in chapter 7.

Appendix A concerns autonomous stochastic differential equations and derives a formula for the *conditional expectation of first passage times.* The conditioning is here on a specified part of the target set being reached before the remainder. Such conditional expectations are natural performance measures for control systems in certain applications. Nevertheless, the material is somewhat peripheral to the robust performance questions which are the main topic of the dissertation; hence it has been placed in appendix.

Appendix B contains a few *technicalities.* These are long but elementary computations needed in proofs in the body of the dissertation. Appendix C contains tables of *frequently used symbols and acronyms.*

The appendices are followed by a bibliography and an index.

1.6 Prerequisites of the reader

Part I in this dissertation assumes that the reader has had some exposure to system theory, linear $\mathcal{H}_\infty$ control and nonlinear deterministic optimal control, e.g. at the level of [67, 128]. Part II assumes in addition some familiarity with stochastic differential equations, e.g. [83].

Part I

Deterministic models

Chapter 2

Multi-dissipative dynamic systems

We consider deterministic dynamic systems with state space representations which are dissipative in the sense of Willems [124] with respect to several supply rates. This property is of interest in robustness analysis and in multi-objective control. We show that under certain assumptions, the dissipated supply rates form a closed convex cone. Furthermore we show convexity and semi-continuity properties of the available storage and required supply as functions of the supply rate.

2.1 Introduction

Dynamic systems which are dissipative in the sense of Willems [124] appear in several areas of control theory. Roughly speaking, a system is dissipative if it is unable to produce a specified quantity, such as energy. The framework is applicable to large-scale systems and robustness problems because dissipativity is preserved under interconnections of systems and because dissipativity for autonomous systems implies stability. Indeed, the framework is a natural extension of Lyapunov theory to input/output systems. Although the notion of dissipativity is a quite general one, most attention has been given to two special cases: passive systems and systems with bounded $\mathcal{L}_2$ gain.

In this chapter we consider deterministic dynamic systems which are dissipative in the sense of [124] with respect to *several* supply rates. Such multi-dissipative systems are interesting from a control perspective for two reasons: It may be a *design objective* that a system should be multi-dissipative, for instance that the closed loop has small gain and that the controller is passive. Secondly, uncertain dynamic elements in the system may be modeled as *multi-dissipative perturbations*. For instance, consider a mechanical system containing two parasitics, each of which is passive and has small $\mathcal{L}_2$ gain. This results in a total of four dissipation properties which the parasitics satisfy together. Such information can be used to show robust stability and performance of the overall system.

Although much literature has been devoted to the topic of systems which are dissipative w.r.t *one* supply rate, it appears that simultaneous dissipation properties have not been studied. In this contribution we show that convexity properties appear nicely when several supply rates are considered at once; for instance, the set of supply rates w.r.t. which a system is dissipative is a convex cone, and for a fixed initial state, the available storage is a convex lower semi-continuous function of the supply rate (see below for definitions and exact statements). These properties are important both from an analytical and a computational point of view.

The chapter is organized as follows: In section 2.2 we summarize some definitions and properties associated with dissipative systems, mostly following [124]. Section 2.3 presents our new results for systems which are dissipative with respect to several supply rates while section 2.4 offers some conclusions.

2.2 Preliminaries

We consider dynamic systems Σ defined by ordinary differential equations in state-space:

$$\Sigma : \quad \begin{array}{rcl} \dot{x}(t) & = & f(x(t), w(t)) \\ z(t) & = & g(x(t), w(t)) \end{array} \tag{2.1}$$

Here, the system has input $w(t) \in \mathbb{W}$, output $z(t) \in \mathbb{Z}$ and state $x(t) \in \mathbb{X}$, and the spaces $\mathbb{X}, \mathbb{W}$ and $\mathbb{Z}$ are Euclidean. We restrict the input signal

$w(\cdot)$ to a signal space $\mathcal{W}$ which is chosen such that the differential equation defines a state transition map $\phi(x, t, w(\cdot))$: If $x(\cdot)$ solves the equation, then $x(t) = \phi(x(0), t, w(\cdot))$.

Associated with the system we have a *supply rate* $s : \mathbb{W} \times \mathbb{Z} \to \mathbb{R}$ which describes a *flow* of some quantity into the system. When the initial state x_0 and the input $w(\cdot)$ is clear from the context we use the shorthand

$$s(t) := s(w(t), g(\phi(x_0, t, w(\cdot)), w(t))) \quad .$$

We do not wish to dwell on technicalities regarding existence, uniqueness and regularity of state trajectories and supplies. We hence simply assume that the input space $\mathcal{W}$ is chosen such that $\phi(x_0, t, w(\cdot))$ is a well defined semi-group, continuous in t, consistent with x_0, causal in $w(\cdot)$ and such that all resulting signals are measurable locally bounded functions[1] of time. Furthermore $\mathcal{W}$ must be closed under switching to guarantee that the principle of optimality holds. These assumptions are for instance met if $f : \mathbb{X} \times \mathbb{W} \to T\mathbb{X}$ is Lipschitz continuous, $g : \mathbb{X} \times \mathbb{W} \to \mathbb{Z}$ and $s : \mathbb{W} \times \mathbb{Z} \to \mathbb{R}$ are locally bounded and measurable, and if $\mathcal{W}$ is the set of piecewise continuous locally bounded signals.

We remark that one could avoid the differential equations all together and define the dynamic system by ϕ, see [124]. One can also define dissipation for input-output systems, see [47].

Our notion of dissipation is the original one of Willems [124]:

Definition 1: A dynamic system Σ is said to be *dissipative* with respect to the supply rate s if there exist a *storage* function $V : \mathbb{X} \to \bar{\mathbb{R}}_+$ such that for all time intervals $[0, T]$, initial conditions x_0 and inputs $w \in \mathcal{W}$ the *dissipation inequality*

$$V(x(T)) \leq V(x(0)) + \int_0^T s(t)\ dt \tag{2.2}$$

holds. □

We use the following formulations interchangeably: Σ is dissipative w.r.t. s; Σ dissipates s; s is dissipated by Σ.

The reader is encouraged to always keep the energy interpretation in mind: s denotes an (abstract) energy flow into the system and V denotes the energy stored in the system.

[1] A function is said to be locally bounded if the image of any bounded set is bounded.

We remark that James has proposed a slightly different definition in [53] where the storage function is required to be locally bounded. It is then possible to restrict attention to lower semi-continuous storage functions which are shown to be exactly the non-negative viscosity solutions in the sense of [23] to the differential formulation of the dissipation inequality

$$\forall w \in \mathbb{W}: \quad V_x(x) f(x,w) \leq s(w, g(x,w)) \tag{2.3}$$

which must hold for all $x \in \mathbb{X}$. The two definitions coincide when the system is locally controllable; then all storage functions are continuous [47, 7].

In many situation it is possible to use the storage function as a Lyapunov function to show various stability properties [124]. For instance, assume that V attains an isolated local minimum at some point x_0 and is continuous in a neighbourhood of x_0 and that $w(\cdot)$ is chosen such that $s(\cdot) \leq 0$, then $x(\cdot) = x_0$ is a Lyapunov stable solution.

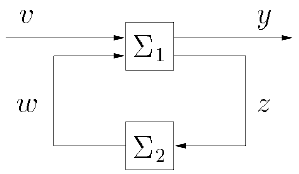

Figure 2.1: Feedback interconnection of dissipative systems

If a collection of dissipative system components are connected in a suitable fashion, then the resulting system will be dissipative as well; as storage function one can use the sum of the storage in each component. This statement seems obvious if one keeps the energy interpretation in mind; however there are a few technical requirements [124]. The simplest such statement is as follows: Assume that the system Σ_1 in figure 2.1 dissipates the supply rate $s_1(v,y) + s_2(w,z)$ and that Σ_2 dissipates $-s_2(w,z)$, then the interconnection (Σ_1, Σ_2), which is a system with input v and output y, dissipates $s_1(v,y)$. Here we have assumed that the interconnection is a well defined dynamic system with a state space representation. In fact a repeated use of this simple statement is sufficient for our purposes.

Sometimes it is useful to consider *strict* dissipation inequalities, i.e. to ask if the system is dissipative w.r.t. $s(w,z) - \alpha(x)$ for some suitable non-negative function α. This is particularly relevant when one is interested in

time constants associated with the system, in robustness w.r.t. perturbations in dynamics or in supply rate, or in stronger stability properties than Lyapunov stability. In particular we follow [99] and say that the system is *strictly output dissipative* w.r.t. the supply rate s iff it is dissipative w.r.t $s - \epsilon|z|^2$ for some constant $\epsilon > 0$. This property is of interest in $\mathcal{L}_2$ stability and performance analysis.

The available storage and the required supply

A dissipative system will in general have many different storage functions for each supply rate, but two are of special interest. First we follow [124] and define the *available* storage

$$V_a(x) = \sup_{w(\cdot),T} \int_0^T -s(t) \; dt$$

where the integral is along the trajectory starting in x and corresponding to $w(\cdot)$. It is easy to see [124] that the available storage is finite everywhere if and only if the system is dissipative, in which case it is in itself a storage function and satisfies $V_a(x) \leq V(x)$ for any other storage function $V(\cdot)$. Furthermore, the available storage has infimum 0 (to see this, let V be a storage function, then so is $V(x) - \inf_\xi V(\xi)$ which implies $V_a(x) \leq V(x) - \inf_\xi V(\xi)$ and hence $\inf_x V_a(x) = 0$). On the other hand, the infimum needs not be attained; consider as an example a system of two electrons moving frictionless in space subject to an external input force u. The supply is the energy delivered by u, the unique storage function is the energy in the system. Minimum storage is found in the limit as the electrons come to rest infinitely far from each other.

Secondly, we define the *required supply* as the least possible supply which can bring the system from a state of minimal available storage to the desired terminal state. More precisely:

$$V_r(x) = \inf_{x(\cdot),w(\cdot),T} \int_0^T s(t) \; dt$$

where the trajectory $x(\cdot)$ must be consistent with $w(\cdot)$ and furthermore satisfy $V_a(x(0)) = 0$ and $x(T) = x$. When no such trajectory exists we define $V_r(x) = \infty$. The required supply satisfies $V_r(x) \geq V(x)$ for any storage function V which has been normalized so that $V(x) = 0$ whenever

$V_a(x) = 0$. Furthermore, if $V_r(x)$ is finite everywhere (i.e. the system is dissipative, there exists at least one point of minimum available storage and any state is reachable from such a point) then $V_r(x)$ is in itself a storage function.

We remark that it is possible to give a more general definition which does not assume the existence of a point of minimum available storage; we shall not pursue this. Also, the reachability assumption will be used frequently in the following. In some situations where it does not hold it may be advantageous to redefine the state space of the system to contain exactly those states which are reachable.

Our definition of the required supply differs slightly from the one of Willems [124]: In this reference infimum is taken over trajectories which start in a fixed, specified point $x(0) = x^*$ with $V_a(x^*) = 0$. In contrast, we allow $x(0)$ to vary as long as $V_a(x(0))$ holds. We believe our definition is more suitable when multiple points of zero available storage exist, for instance several equilibrium points. A consequence of our definition is that any x which satisfies $V_a(x) = 0$ also satisfies $V_r(x) = 0$.

Sometimes we use the notation $V_a(x, s)$ and $V_r(x, s)$ to stress which supply rate we are referring to. We remark that the availabe storage and the required supply are viscosity solutions to differential dissipation *equalities* corresponding to the inequality (2.3), provided that they are continuous and under certain assumptions [7]. See also the example in section 5.7 on page 116 below.

2.3 Properties of multi-dissipative dynamic systems

In this section we consider a system Σ of the form (2.1) which is dissipative with respect to more than one supply rate. We investigate the set of supply rates which are dissipated by the system and we show several properties which are related to the convexity of this set.

It was noted already in [124] that the storage functions for a dynamical system with respect to a single supply rate form a convex set. For multi-dissipative systems this fact extends easily to the following:

Proposition 2: Let $\mathcal{V}$ be a linear space of functions $\mathbb{X} \to \mathbb{R}$ and let $\mathcal{S}$ be a linear space of supply rates $\mathbb{W} \times \mathbb{Z} \to \mathbb{R}$. Then those pairs (V, s) for

which V is a storage function w.r.t. the supply rate s form a convex cone, i.e.

$$\{(V, s) \in \mathcal{V} \times \mathcal{S} \mid V \geq 0 \text{ and } (V, s) \text{ satisfy the dissipation inequality (2.2) }\}$$

is a convex cone. Furthermore this set is closed with respect to pointwise (in $\mathbb{X}$) convergence of storage functions V and local uniform convergence (over $\mathbb{W} \times \mathbb{Z}$) of supply rates s. △

Regarding the last closedness statement, one could have considered several different topologies on the space $\mathcal{S}$ of supply rates. Throughout, we shall restrict attention to the topology corresponding to local uniform convergence over $\mathbb{W} \times \mathbb{Z}$; this mode of convergence appears to be most useful in applications. We recall the standard definition:

Definition 3: We say that $s_i \to s$ locally uniformly if, for every compact subset Ω of $\mathbb{W} \times \mathbb{Z}$ and every $\epsilon > 0$, there exists an $N > 0$ such that $\sup_{(w,z)\in\Omega} |s_i(w, z) - s(w, z)| < \epsilon$ for $i > N$. □

We remark that if $s_i \to s$ locally uniformly, then $\sup_{t\in[0,T]} |s_i(t) - s(t)| \to 0$ for any finite T and any trajectory which satify our standing assumption that all signals are locally bounded functions of time.

The proof of proposition 2 is a quite straightforward exercise of the machinery of dissipation theory; we include it for the convenience of the reader.

Proof: [of the proposition] Too see that the set is a convex cone, let the system be dissipative w.r.t. the supply rates s_1 and s_2 with storage functions V_1 and V_2, respectively. We must then show that $\lambda_1 V_1 + \lambda_2 V_2$ is a storage function with respect to $\lambda_1 s_1 + \lambda_2 s_2$ for any $\lambda_1, \lambda_2 \geq 0$. To this end, let the initial state $x(0)$, the input $w(\cdot)$ and the final time T be arbitrary; then the dissipation inequalities

$$V_i(x(T)) \leq V_i(x(0)) + \int_0^T s_i(t)\ dt$$

hold for $i = 1, 2$. Multiply these inequalities with $\lambda_1, \lambda_2 \geq 0$ and add the two to obtain

$$\sum_{i=1}^{2} \lambda_i V_i(x(T)) \leq \sum_{i=1}^{2} \lambda_i V_i(x(0)) + \int_0^T \sum_{i=1}^{2} \lambda_i s_i(t)\ dt$$

which says that $\lambda_1 V_1 + \lambda_2 V_2$ is a storage function with respect to $\lambda_1 s_1 + \lambda_2 s_2$.

To see that the set is closed, let $V_i \to V$ pointwise in $\mathbb{X}$ and let $s_i \to s$ locally uniformly over $\mathbb{W} \times \mathbb{Z}$. Consider an arbitrary trajectory such that

$$V_i(x(T)) \leq V_i(x(0)) + \int_0^T s_i(t)\ dt \quad .$$

Then we have $\int_0^T s_i(t)\ dt \to \int_0^T s(t)\ dt$ due to local uniform convergence of s_i since all signals by assumption are bounded on the bounded interval $[0, T]$. Combining with pointwise convergence of $V_i(x(\cdot))$ we get

$$V(x(T)) \leq V(x(0)) + \int_0^T s(t)\ dt$$

which should be shown. ■

We see from proposition 2 that if the system dissipates any supply rate in a given set $\mathbb{S} \subset \mathcal{S}$, then it is dissipates any supply rate in the convex conic hull of $\mathbb{S}$. This was also noted in [45].

An interesting question is if the set of dissipated supply rates is closed under some given topology on the space $\mathcal{S}$ of supply rates. For instance, in $\mathcal{L}_2$-gain analysis one considers supply rates $s_\gamma(w, z) = \gamma^2|w|^2 - |z|^2$ and define the $\mathcal{L}_2$-gain γ^* as the infimum over all numbers $\gamma > 0$ such that the system is dissipative w.r.t. s_γ. The question if the system is dissipative w.r.t. s_{γ^*} hence arises naturally. In this case it is [120], but the question has not been considered for more general families of supply rates. Notice that the closedness shown in proposition 2 does not answer this question.

A first result in this direction is obtained with the notion of a cyclo-dissipative system:

Definition 4: The system Σ is cyclo-dissipative w.r.t. the supply rate s if

$$\int_0^T s(t)\ dt \geq 0$$

for any T and any pair $w(\cdot), x(\cdot)$ such that $x(0) = x(T)$. □

This definition deviates slightly from the one in [47] where the inequality is required to hold only when $x(0) = x(T) = 0$; here, we have no reason to discriminate the state $x = 0$. A dissipative system is obviously cyclo-dissipative whereas the converse implication does not hold in general [47]. We can now pose the result:

Proposition 5: Assume that Σ is cyclo-dissipative w.r.t. s_i, $i \in \mathbb{N}$, and that $s_i \to s$ locally uniformly as $i \to \infty$. Then Σ is cyclo-dissipative w.r.t. s. △

Proof: Let $w(\cdot), x(\cdot)$ be any trajectory such that $x(0) = x(T)$, let $z(\cdot)$ be the corresponding output. Since all signals by assumption are locally bounded, there exists a bounded set $\Omega \in \mathbb{W} \times \mathbb{Z}$ such that $(w(t), z(t)) \in \Omega$ for $t \in [0, T]$. Let $\epsilon > 0$ be arbitrary and let i be sufficiently large such that $\sup_{(w,z)\in\Omega} |s_i(w, z) - s(w, z)| < \epsilon$. Then

$$\int_0^T s(t)\ dt \geq \int_0^T s_i(t)\ dt - \epsilon T \geq -\epsilon T$$

since the system is cyclo-dissipative w.r.t. s_i. Letting $\epsilon \to 0$ yields the desired conclusion. ■

It follows that the set of cyclo-dissipated supply rates is a closed convex cone. An appealing conjecture is that the same statement holds if one replaces the word cyclo-dissipated with dissipated. This is not the case, however, as the following example demonstrates.

Example 6: Consider a scalar integrator, i.e. a system with state space $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = w, \quad z = x$$

and let the space $\mathcal{S}$ of supply rates be the span of the two rates wz and z^3w. Consider a sequence of supply rates

$$s_i(w, z) = -2wz + \frac{4}{i} z^3 w \quad .$$

It is then easy to see that the system is dissipative w.r.t. s_i; the available storage is

$$V_a(x, s_i) = -x^2 + \frac{1}{i} x^4 + \frac{i}{4} \quad .$$

In fact the dissipation inequalities always hold with equality (in the terminology of [124] the system is *lossless* w.r.t. s_i). The supply rates s_i converge locally uniformly to $s(w, z) = -2wz$ and it is easy to see that the system does not dissipate s: For the dissipation inequality to be satisfied the storage function must necessarily be in the form $V(x) = -x^2 + K$ and no K exists such that V is non-negative. However, the system is cyclo-dissipative w.r.t. s in accordance with the previous result. □

In order to get the desired result we need an additional assumption on the states of zero available storage:

Proposition 7: Let s_i, $i \in \mathbb{N}$, be a sequence of dissipated supply rates which converges locally uniformly to the supply rate s. Assume that the set of minimal available storage $\{x \mid V_a(x, s_i) = 0\}$ is independent of $i \in \mathbb{N}$ and non-empty, and that the entire state space $\mathbb{X}$ is reachable from this set. Then the system is dissipative w.r.t. s. $\triangle$

Proof: Consider an arbitrary trajectory such that $V_a(x(0), s_i) = 0$ and define

$$J := \int_0^T s(t)\ dt$$

where $T > 0$ is arbitrary. Let $\epsilon > 0$ be arbitrary and choose i sufficiently large such that $|s_i(t) - s(t)| < \epsilon$ for $t \in [0, T]$; this is possible since all signals are bounded on $[0, T]$ and $s_i \to s$ locally uniformly on $\mathbb{W} \times \mathbb{Z}$. It follows that

$$0 \leq \int_0^T s_i(t)\ dt \leq J + \epsilon T$$

where the first inequality holds because the trajectory starts in a point of zero available storage w.r.t. s_i. Since $\epsilon > 0$ was arbitrary we conclude that $J \geq 0$.

Now consider any continuation of the trajectory starting at time T in the state $x(T)$ and ending at time $T' > T$. Repeating the above argument we see that

$$\int_0^{T'} s(t)\ dt \geq 0$$

which in turn implies that

$$\int_T^{T'} -s(t)\ dt \leq J \quad .$$

We conclude that $V_a(x(T), s) \leq J < \infty$. Now notice that the point $x(T)$ can be chosen arbitrarily since the entire state space is reachable; it follows that the available storage w.r.t. s is finite everywhere. We conclude that the system is dissipative w.r.t. s. $\blacksquare$

The hypothesis that the set of zero initial storage is independent of i fails in example 6 above. In this example we have $V_a(x, s_i) = 0 \Leftrightarrow |x| = \sqrt{i/2}$.

In many applications there is only one set which can be a set of minimal available storage, for instance a single zero-input equilibrium point. In these situations we conclude that the dissipated supply rates form a closed convex cone.

One can also derive closedness properties using theory for partial differential equations, rather than system theory, for instance following [23, 53]. We point out that in comparison with this approach, proposition 7 has the strength of not imposing local boundedness, continuity, or other regularity requirements on the storage functions.

The previous results clarifies the structure of the set of dissipated supply rates. We now turn to the properties of the available storage and required supply, seen as functions of the supply rate.

Proposition 8: Let $\mathbb{S}$ be a convex set of dissipated supply rates and let $x \in \mathbb{X}$ be fixed. Then $V_a(x, s)$ is a convex lower semi-continuous function of $s \in \mathbb{S}$. If furthermore the set $\{x | V_a(x, s) = 0\}$ is independent of $s \in \mathbb{S}$ and non-empty, and if the entire state space is reachable from this set, then $V_r(x, s)$ is a concave upper semi-continuous function of $s \in \mathbb{S}$. $\triangle$

Proof: First we show that $V_a(x, \cdot)$ is convex in the supply rate: Fix the initial condition $x(0)$ and define the functional J_a on $\mathcal{W} \times \mathbb{R}_+ \times \mathcal{S}$ by

$$J_a(w(\cdot), T, s) = \int_0^T -s(t)\ dt$$

where the integrand is evaluated along the trajectory starting in $x(0)$ and corresponding to $w(\cdot)$. Notice that J_a is convex in s; even linear. Hence

$$V_a(x(0), s) = \sup_{w(\cdot), T} J_a(w(\cdot), T, s)$$

is also convex since the supremeum of any family of convex functionals is convex.

Next we show that $V_a(x, \cdot)$ is lower semi-continuous: Let $s \in \mathbb{S}$ and let s_i be a sequence in $\mathbb{S}$ which converges locally uniformly to s; we must then show that

$$\liminf_{i \to \infty} V_a(x, s_i) \geq V_a(x, s) \quad .$$

Choose $\epsilon > 0$ and let $x(\cdot)$ be a trajectory with $x(0) = x$ such that

$$\int_0^T -s(t)\ dt \geq V_a(x, s) - \epsilon \quad .$$

Now choose i sufficiently large such that $|s_i(t) - s(t)| < \epsilon/T$ on $[0,T]$; then

$$\int_0^T -s_i(t)\ dt \geq \int_0^T -s(t)\ dt - \epsilon \geq V_a(x,s) - 2\epsilon$$

which implies that $V_a(x,s_i) \geq V_a(x,s) - 2\epsilon$ for i sufficiently large. Now let $\epsilon \to 0$ to obtain the desired conclusion.

To show concavity of $V_r(\bar{x},\cdot)$ under the additional assumptions, we follow the argument above: Let Ω denote those $(w(\cdot), x_0, T)$ in $\mathcal{W} \times \mathbb{X} \times \bar{\mathbb{R}}_+$ for which $V_a(x_0, s) = 0$ for $s \in \mathbb{S}$, and for which the trajectory starting in x_0 and corresponding to $w(\cdot)$ satisfies $x(T) = \bar{x}$. Now define the functional J_r on $\Omega \times \mathcal{S}$ by

$$J_r(w(\cdot), x_0, T, s) = \int_0^T s(t)\ dt$$

which is concave, in fact linear, in s. Now notice that $V_r(\bar{x}, s)$ is the infimum of J_r over the set Ω and hence concave.

Finally we show upper semi-continuity of $V_r(x,\cdot)$. Choose $\epsilon > 0$ and let $x(\cdot)$ be a trajectory which starts with zero available storage, i.e. $V_a(x(0), s) = 0$, ends in $x(T) = x$, and which satisfies

$$\int_0^T s(t)\ dt \leq V_r(x,s) + \epsilon \quad .$$

Now choose i sufficiently large such that $|s_i(t) - s(t)| < \epsilon/T$ on $[0,T]$, then

$$\int_0^T s_i(t)\ dt \leq \int_0^T s(t)\ dt + \epsilon \leq V_r(x,s) + 2\epsilon$$

which implies that $V_r(x,s_i) \leq V_r(x,s) + 2\epsilon$ for i sufficiently large. Again let $\epsilon \to 0$ to obtain the desired result. ■

With this result in mind it is natural to ask if the available storage is also an upper semi-continuous function of the supply rate, and thus continuous. In general, the answer to this question is negative:

Example 9: Consider the autonomous system with state space $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = -x, \quad z = x.$$

Let a sequence of supply rates s_i be given by

$$s_i(z) = \begin{cases} -\frac{1}{i} & \text{if } 2^{-i(i+1)/2} \leq z \leq 2^{-(i-1)i/2} \\ 0 & \text{else.} \end{cases}$$

The space $\mathcal{S}$ is the linear span of these s_i for $i \in \mathbb{N}$, and we take $\mathbb{S} = \mathcal{S}$. It is then straightforward to see that

$$V_a(x, s_i) = \log 2$$

for any $x \geq 1$. On the other hand, the supply rates s_i converge uniformly to the supply rate $s = 0$ for which $V_a(x,0) = 0$. Hence the available storage $V_a(x,\cdot)$ is not an upper semi-continuous function of the supply rate. □

However, an example of particular interest is when the set $\mathbb{S}$ of dissipated supply rates is a convex polytope, i.e. the convex hull of a finite collection of supply rates. In this situation upper semi-continuity follows from convexity:

Corollary 10: Take the same assumptions as in proposition 8 and assume in addition that $\mathbb{S}$ is a convex polytope. Then $V_a(x,\cdot)$ and $V_r(x,\cdot)$ are continuous functions of $s \in \mathbb{S}$. □

Proof: The statement follows from a standard result [92, p. 84] according to which a convex function defined on a convex polytope is upper semi-continuous. ■

Another situation where continuity follows is when the available storage and the required supply coincide. This is the case for lossless systems under certain assumptions, see [124].

We summarize and illustrate the discussion with the following simple example concerning $\mathcal{L}_2$-gain analsysis of a scalar linear system.

Example 11: Consider the system

$$\dot{x} = -x + w, \quad z = x$$

and the two supply rates $s_1 = |w|^2$ and $s_2 = -|z|^2$ corresponding to an analysis of $\mathcal{L}_2$-gain from w to z. Let the space $\mathcal{S}$ of supply rates be the span of s_1 and s_2.

Since the system is linear and the supply rates are quadratic we know [124] that if the system is dissipative w.r.t. the rate $\lambda_1 s_1 + \lambda_2 s_2$ then there exist a quadratic storage function $V(x) = \alpha x^2$. The differential dissipation inequality then reduces to the linear matrix inequality

$$\begin{bmatrix} -2\alpha + \lambda_2 & \alpha \\ \alpha & -\lambda_1 \end{bmatrix} \leq 0 \quad .$$

The set of those $\alpha, \lambda_1, \lambda_2$, for which the linear matrix inequality holds, is a cone. Let us concentrate on the subcone for which $\lambda_2 > 0$. We may obtain a cross-section of this cone by fixing $\lambda_2 = 1$ and examine which values α and λ_1 result in a supply rate and a storage function which satisfy the dissipation inequality. A little manipulation yields that this set is characterized as

$$\alpha \geq \frac{1}{2}, \qquad (2\alpha - 1)\lambda_1 \geq \alpha^2 \quad .$$

This set is depicted in figure 2.2. It has the structure which was predicted by the previous results: It is convex and closed as is its projection on the λ_1-axis. Furthermore, the available storage and the required supply are continuous functions of λ_1, convex and concave, respectively. In addition the set has the special feature of being unbounded since s_1 is sign definite. □

2.4 Chapter conclusion

For a dissipative dynamic system, we have asked the question: With respect to which supply rates is the system dissipative? We have shown elementary properties associated with these *dissipated* supply rates: They form a convex cone which is also closed under additional assumptions. Furthermore we have investigated continuity properties of the available storage and the required supply, seen as functions of the supply rate. For the important special case of convex polytopes of supply rates, we have shown that these functions are continuous.

Many of our results have been shown under the assumption that the sets of zero available storage are independent of the supply rate under consideration. An interesting topic of future research would be to relax this assumption.

Our original motivation for this study was the situation where a dynamic system contains perturbations which are known to be multi-dissipative. In this situation the inherent convexity can be employed to obtain quite sharp conditions for robust stability and performance by means of convex optimization: we optimize over the set of supply rates with respect to which the nominal system is dissipative. In the special case of linear systems and quadratic supply rates the dissipation inequalities are linear matrix inequalities and the numerical tool is semidefinite programming. Some results along these lines follow in chapter 3 below.

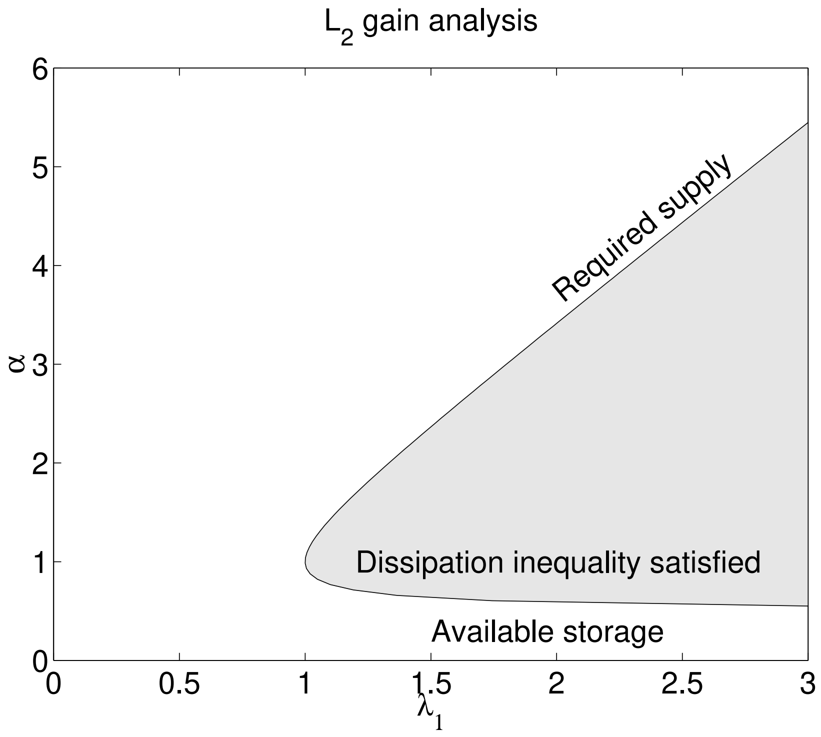

Figure 2.2: A cross-section of the cone of supply rates and storage functions which satisfy the dissipation inequality.

Another application of the theory of multi-dissipative systems is the problem of *control for multi-dissipation*, i.e. find a controller in some class such that the resulting closed-loop system is dissipative w.r.t. a family of supply rates. For instance, one may require that the closed loop is small gain and that the controller is passive. In chapter 4 below, which concerns the problem of simultaneous $\mathcal{H}_\infty$ control, such a problem of control for multi-dissipation arises.

Chapter 3

Robustness towards multi-dissipative perturbations

We investigate the robustness of an interconnection of a nominal system, described by nonlinear ordinary differential equations, and an unknown perturbation which is dissipative with respect to several supply rates. We give sufficient conditions for global robust stability and performance in terms of existence of solutions to nonlinear partial differential inequalities of the Hamilton-Jacobi-Bellman type with certain extra degrees of freedom, namely a vector of weights. We then specialize to linear systems with quadratic supply rates where the analysis reduces to linear matrix inequality problems.

It is popular to deal with uncertainty in control problems using the framework of dissipation (in the sense of Willems [124] and the previous chapter) because dissipativity is preserved under interconnections of systems and because dissipativity for autonomous systems implies stability. This makes it practical to model uncertainty by dissipative perturbations, and to pose as design specification that the overall system is dissipative. A common example of a dissipation property is bounded $\mathcal{L}_2$-gain. This particular property leads to linear or nonlinear $\mathcal{H}_\infty$ control, where the uncertainty is modelled by perturbations which have bounded $\mathcal{L}_2$-gain, and where performance of the overall system is measured by its $\mathcal{L}_2$-gain as well. Also passive pertubations are common; for instance stability proofs of certain adaptive control systems employ passivity-based arguments.

In this chapter we consider robustness towards deterministic dynamic perturbations which are dissipative with respect to *several* supply rates. Section 3.1 motivates this problem by providing examples of such multi-dissipative perturbations. In this section we also compare the framework with that of integral quadratic constraints. In section 3.2, we demonstrate that information regarding multiple dissipation properties of the perturbations can be included in an robustness analysis in an operational fashion. The resulting conditions on the nominal part of the system are partial differential inequalities of the Hamilton-Jacobi-Bellman type with certain extra degrees of freedom. Section 3.3 specializes the discussion to linear systems and quadratic supply rates; in these situations linear matrix inequalities becomes an efficient numerical tool with which we can also address related problems involving parameter uncertainty, or of robust $\mathcal{H}_2$ performance. Finally, section 3.4 offers some concluding remarks and points out a number of open problems.

3.1 Multi-dissipative perturbations

The aim of this section is to provide a few examples of multiple dissipation properties of perturbations in control systems. The section merely summarizes some ideas - some well known, others seemingly new - and does not present new results.

For a single dynamic perturbation $w(\cdot) = \Delta z(\cdot)$, typical dissipation properties are related to gain and phase properties. For instance, linear positive real perturbations - or more generally nonlinear passive perturbations - are dissipative w.r.t. the supply rate $s(w,z) = \langle w, z\rangle$. Similarly Δ has $\mathcal{L}_2$-gain (or $\mathcal{H}_\infty$ norm) less than or equal to $\gamma > 0$ if and only if Δ is dissipative w.r.t. the supply rate $s(w,z) = \gamma^2|z|^2 - |w|^2$ - this can be generalized to any $\mathcal{L}_p$ induced norm for finite p.

When the perturbation represents parasitic dynamics, for instance oscillatory modes in a mechanical or electrical system, the passivity follows from

the fact that such oscillations cannot produce physical energy. More generally, physical conservation laws give rise to dissipation properties. Conservation of mass, volume, free thermodynamic energy, or momentum can be cast as dissipation properties.

Bounds on static (memoryless) nonlinearities can also be expressed in terms of dissipation properties, although the information that Δ is static is lost. Specifically, let $w(t)$ and $z(t)$ be scalar and let Δ be given by

$$w(t) = (\Delta z)(t) = \phi(z(t))$$

where $\phi:\ \mathbb{R} \to \mathbb{R}$ is known to satisfy the inequality $\psi(z,\phi(z)) \geq 0$, then obviously Δ is dissipative w.r.t. the supply rate $s(w,z) = \psi(z,w)$. A particular popular class of bounds are the linear *sector bounds* which are common in the field of absolute stability, see [59] and the references therein. For instance, if the graph of ϕ lies between the lines $w = az$ and $w = bz$ for known real numbers $a < b$ then the corresponding function ψ may be taken as the quadratic form

$$\psi(z,w) = (w\ z)\begin{bmatrix} -1 & \frac{a+b}{2} \\ \frac{a+b}{2} & -ab \end{bmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \quad . \tag{3.1}$$

It is important to examine how much conservativeness one introduces by neglecting that the perturbation is static. When the supply rate s is quadratic, a partial answer to this question is obtained by examining which linear time invariant systems dissipate s.

The above examples illustrate how one may establish *single* dissipation properties of perturbations. Our prime example of a multi-dissipative dynamic perturbation concerns parasitic dynamics which are bounded and passive:

Example 12: [Modelling of multi-dissipative perturbations] Consider the spring-mass system in figure 3.1, which is a simple model of a one-dimensional position regulator system. The force u is the output of an linear time invariant controller. We consider the small mass as an unmodelled parasitic, and the parameters associated with it to be very uncertain.

The overall interconnection of the small mass and the remaining system may be written in the form of figure 3.2. The error signal z is then the velocity $\dot{y}$ of the large mass while the disturbance w is the force acting from small mass on the large mass. With this formulation, Δ is given by

$$\Delta(s) = \frac{(k+cs)ms}{ms^2+cs+k} \quad .$$

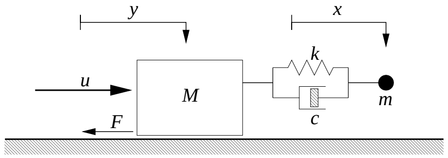

Figure 3.1: A position regulator with parasitic dynamics

The transfer function Δ is positive real, i.e. dissipative w.r.t. the supply rate $-s_1(z,w) = zw$, since this supply rate corresponds to the mechanical energy supplied to the parasitic. It is also small gain, i.e. dissipative w.r.t. the supply rate $-s_2(z,w) = \gamma^2 z^2 - w^2$ for $\gamma \geq \|\Delta\|_\infty \approx km/c\sqrt{1+c^2/km}$.

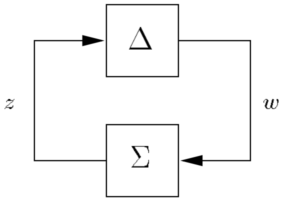

Figure 3.2: System and perturbation in feed-back

One can easily imagine situations where physical considerations or a few simple experiments provide a reasonable bound on γ but where we nevertheless do not wish to estimate m, c and k. Indeed, in many situations we do not wish to specify the order of Δ. In these situations, the only information about Δ we wish to make use of in the subsequent analysis is the two dissipation properties. □

See also [33] for a discussion of this example in the context of integral quadratic constraints.

Another way multiple dissipation properties arise is when the input z and the output w to the perturbation Δ can be partitioned as

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}$$

and the perturbation Δ is block diagonal, i.e.

$$w = \Delta z \iff w_i = \Delta_i z_i, \; i = 1, \ldots p \quad .$$

This block structure occurs when each perturbation Δ_i is associated with a different physical component in the system (if each block diagonal element Δ_i is bounded in $\mathcal{H}_\infty$-norm, then the resulting problem is μ-analysis). It is clear that the total perturbation Δ inherits the dissipation properties of each perturbation element Δ_i. This may quickly lead to a quite large number of dissipation properties of Δ as a realistic control problem typically will contain uncertain elements in many different places in the control loop.

We conclude this section with an example demonstrating how one may establish necessary and sufficient conditions for the robustness of a system containing a multi-dissipative perturbation:

Example 13: [A non-conservative robustness condition] Continuing example 12 above, the suitable analysis question is: When is Σ robustly stable towards perturbations Δ which are linear time invariant and dissipative with respect to $-s_i$ for $i \in \{1,2\}$? Here we construct necessary and sufficient conditions through frequency domain considerations. First, rescale the system such that that $\gamma = 1$. The requirement that Δ is linear time invariant, passive and small gain then is equivalent to the transfer function $\Delta(s)$ mapping the closed right half of the complex plane into the set A in figure 3.3, i.e.

$$\forall s \in \bar{\mathbb{C}}^+ : \; \Delta(s) \in A = \{s \in \mathbb{C} \mid \mathbf{Re}\; s \geq 0 \wedge |s| \leq 1\} \tag{3.2}$$

The interconnection is unstable if and only if the closed loop has a pole in the closed right half plane, i.e. there exists an $s \in \bar{\mathbb{C}}^+$ such that

$$\Sigma(s)\Delta(s) = 1 \quad .$$

So the interconnection is stable *for all* Δ such that (3.2) holds if and only if Σ maps the closed right half plane into the region B in figure 3.3, i.e.

$$\begin{aligned} \forall s \in \bar{\mathbb{C}}^+ : \; \Sigma(s) \in B &:= \{s \in \mathbb{C} \mid \frac{1}{s} \notin A\} \qquad (3.3) \\ &= \{s \in \mathbb{C} \mid \mathbf{Re} s < 0 \vee |s| < 1\} \quad . \end{aligned}$$

An alternative characterization of the set B is

$$B = \{s \in \mathbb{C} \mid \exists \alpha > 0 : \; |s + \alpha| < |i + \alpha|\} \quad .$$

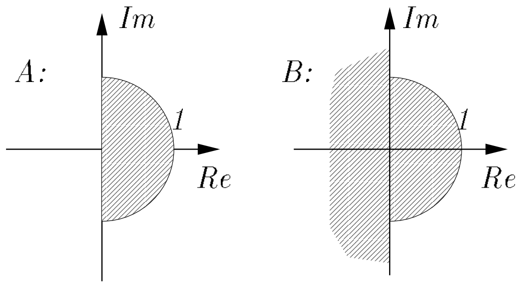

Figure 3.3: The subsets A and B of the complex plane

So the condition (3.3) holds if and only if there exists a map $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}^+$ such that

$$\forall s \in \bar{\mathbb{C}}^+ \; : \; |\Sigma(s) + \alpha(s)| < |i + \alpha(s)| \quad .$$

The inequality in this expression can be restated as

$$(1 \; \bar{\Sigma}(s)) \begin{bmatrix} 1 & -\alpha(s) \\ -\alpha(s) & -1 \end{bmatrix} \begin{pmatrix} 1 \\ \Sigma(s) \end{pmatrix} > 0 \quad . \tag{3.4}$$

To recapitulate, the feedback system is stable if and only if there exists an $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}^+$ such that this holds for all $s \in \bar{\mathbb{C}}^+$. This is exactly the type of stability conditions that appear in [57] (see also the references therein); in the nomenclature used there Δ satisfies two *integral quadratic constraints* (IQC's). □

In more complicated situations, involving several perturbations or non-linear systems, the problem of obtaining non-conservative conditions for robustness is still untractable. In the remainder of this chapter we derive *sufficient* conditions only. We shall later return to this example in order to comment on the conservativeness inherent in our conditions.

3.2 Robustness analysis

We now turn to the interconnection of figure 3.4 where Σ is the nominal system, Δ is a multi-dissipative perturbation and v is an exogenous deterministic signal. Throughout the section, x denotes a state of Σ while ξ denotes a state of Δ.

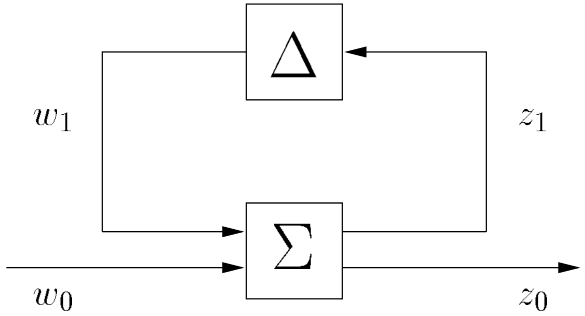

Figure 3.4: Setup for robust performance analysis.

The problems we consider are generalizations of robust non-linear $\mathcal{H}_\infty$ analysis problems. Several versions of these problems exist; one is the following: The unknown perturbation Δ is a causal system with $\mathcal{L}_2$-gain less than or equal to 1, i.e. is dissipative w.r.t. the supply rate $|z|^2 - |w|^2$. Commonly, the interconnection is assumed to be at rest at the initial time $t = 0$. The aim of the analysis is to establish an upper bound for

$$\int_0^T |y(t)|^2 \; dt$$

which holds for all perturbations Δ, all final times T and all inputs $v(\cdot)$ with $\int_0^T |v|^2 \; dt \leq 1$.

The aim of this section is to consider robustness analysis problems which generalize this $\mathcal{H}_\infty$ problem above in several directions. The objective is to establish a bound on

$$\int_0^T l(y(t)) \; dt$$

where l is a given non-negative *running cost.* To retrieve robust $\mathcal{H}_\infty$ analysis, use $l(y) = |y|^2$.

The following list makes precise the assumptions under which we obtain our robustness result:

Assumption 14:

1. The interconnection of Σ and Δ is well posed in the following sense: To each initial conditions x_0 and ξ_0 and each input $v(\cdot)$ correspond

unique state trajectories $x(\cdot)$ and $\xi(\cdot)$ which are continuous and depend causally on $v(\cdot)$, and the signals $w(\cdot)$, $z(\cdot)$ are measurable and locally bounded[1] functions of time, at least up to some finite escape time.

2. The perturbation Δ is dissipative w.r.t. to the p measurable and locally bounded supply rates $-s_i$, $i = 1, \ldots, p$, with available storage $V_a(\xi_0, -s_i) \leq \beta_i$ for a known set of bounds β_i. Here ξ_0 is the initial state of the perturbation Δ.

3. The input $v(\cdot)$ satisfies

$$\int_0^T s_v(t)\, dt \leq \beta_v$$

for a known bound β_v and any time $T > 0$, provided that no finite escape time occurs before T. Here $s_v(v, y)$ is a given measurable and locally bounded supply rate.

□

The motivation behind the first assumption is as follows: We disregard pathological situations where non-unique or discontinuous state trajectories occur, but we do not wish to exclude a finite escape time *a priori*; rather we wish to establish conditions under which a finite escape time cannot occur.

Regarding the second assumption, we retrieve robust $\mathcal{H}_\infty$ analysis by choosing $p = 1$, $s_1 = |w|^2 - |z|^2$, and $\beta_1 = 0$. In applications, it is not always reasonable to assume zero initial storage in the perturbations (i.e, $\beta_i = 0$) as is done in the robust $\mathcal{H}_\infty$ problem as outlined here; in order to study robustness of transient behaviour it is essential to allow some bounded amount of initial storage in the perturbation. (If focus is on stability or steady-state behavior assumptions corresponding to zero initial storage may be reasonable and are seen in the IQC literature, e.g. [77, 58, 33]; an exception is [96]). On the other hand, the assumption that the perturbations have bounded initial storage is often quite reasonable - although it may be difficult to establish the exact size of these bounds. A similar discussion applies to the assumption that the input v has a bounded resource given by the rate s_v and the bound β_v.

[1] A function is said to be locally bounded if the image of any bounded set is bounded.

With this problem setup the sufficient condition for our objectives to be met is that the nominal system is dissipative w.r.t. some supply rate which matches the rates s_i, s_v and the running cost l. More precisely we can state the following theorem:

Theorem 15: [Robustness implications of dissipativity] Let assumption 14 hold and assume in addition that the nominal system Σ is dissipative w.r.t. the supply rate $\sum_i d_i s_i + d_v s_v - l$ for some non-negative weights d_i, d_v. Let V be a corresponding storage function. Then the following holds:

1. If no finite escape time occurs, then the interconnection is dissipative w.r.t. the supply rate $d_v s_v - l$.

2. The state $x(T)$ remains in the set

$$\{x \mid V(x) \leq V(x_0) + \sum_i d_i \beta_i + d_v \beta_v\}$$

for any $T > 0$ such that no finite escape time occurs before T.

3. If $V(\cdot)$ and $\sum_i d_i V_a(\cdot, s_i)$ are proper[2] functions, then no finite escape time occurs.

4. The performance bound

$$\int_0^T l\, dt \leq V(x_0) + \sum_i d_i \beta_i + d_v \beta_v$$

holds for any $T > 0$ such that no finite escape time occurs before T.

□

Proof: Fix the initial states x_0 and ξ_0 and the input v and let $T > 0$ be a time such that no finite escape time occurs before T. As candidate storage function for the interconnection w.r.t. the supply rate $d_v s_v - l$ we take $W(x, \xi) = V(x) + \sum_i d_i V_a(\xi, -s_i)$. It is then easy to see that W satisfies the dissipation inequality which proves item 1. Using the non-negativeness

[2] A real-valued function is said to be proper iff all preimages $V^{-1}(I)$ of bounded intervals $I \subset \mathbb{R}$ are bounded.

of $V_a(\xi, -s_i)$ and of l we get

$$\begin{aligned} V(x(T)) &\leq W(x(T), \xi(T)) \\ &\leq W(x_0, \xi_0) + \int_0^T d_v s_v - l \; dt \\ &\leq V(x_0) + \sum_i d_i \beta_i + d_v \beta_v \end{aligned}$$

as claimed in item 2. If furthermore V and $\sum_i d_i V_a(\cdot, -s_i)$ are proper then this implies that $x(t)$ and $\xi(t)$ remain in a fixed bounded set which excludes the existence of a finite escape time and hence proves item 3. Finally item 4 is simply a rearrangement of the dissipation inequality of item 1. ■

A key feature of the theorem is that the characterization is *convex*: The set of those storage functions V and weights d_i, d_v which satisfy the dissipation inequality is convex (proposition 2 on page 32 above). Furthermore, if we wish to search for the *best* weights d_i, d_v, i.e. those that lead to smallest available storage in a fixed initial point, then this involves minimizing a convex continuous function (proposition 8 on page 37 above and the subsequent corollary 10).

Another feature of the theorem is that it simultaneously addresses robust stability and performance: Robust performance in the sense of a bound on an integral is given in item 4. To demonstrate that item 2 can be used to show robust stability, we first establish a useful lemma:

Lemma 16: [Bounding the state trajectory] Let $\Omega \subset \mathbb{X}$ be an open set and let $x(t)$, $t \geq 0$, be a state trajectory such that $x(0) \in \Omega$. Let $\gamma > 0$ be such that $V(x(t)) \leq \gamma$ at least until $x(t)$ leaves Ω. Let $\mathbb{A}$ be the largest connected subset of $V^{-1}([0, \gamma]) \cap \Omega$ which contains the initial state $x(0)$. Assume that $\mathbb{A}$ is compact. Then $x(t)$ remains in $\mathbb{A}$ for $t \in [0, \infty)$. □

Proof: Assume that $x(t)$ leaves Ω in finite time. Let t_2 denote the time of first exit from Ω and let t_1 denote the last preceding time of exit from $\mathbb{A}$. Since $\mathbb{A}$ is closed and Ω is open we have $t_1 < t_2$. Let $t \in (t_1, t_2)$ and define $\mathbb{B} = \mathbb{A} \cup \{x(\tau) \; : \; \tau \in [t_1, t]\}$. Then $\mathbb{B}$ contains x_0 and is a connected subset of $V^{-1}([0, \gamma]) \cap \Omega$ of which $\mathbb{A}$ is a strict subset. This is in contradiction with the definition of $\mathbb{A}$. We conclude that $x(t)$ remains in Ω, hence also in $\mathbb{A}$, until a finite escape time. Since $\mathbb{A}$ is bounded this excludes finite escape times; hence $x(t)$ always remains in $\mathbb{A}$. ■

The importance of $\mathbb{A}$ being closed is illustrated in figure 3.5. Here $\mathbb{A}$ is not

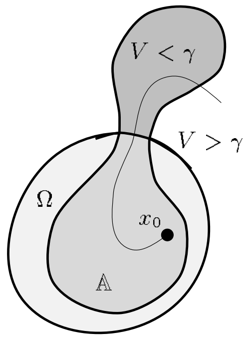

Figure 3.5: A pre-image $\mathbb{A} = V^{-1}([0, \gamma]) \subset \Omega$ which is not closed.

closed and hence the state trajectory can leave $\mathbb{A}$ and Ω simultaneously; once the state has exited Ω the bound $V(x) \leq \gamma$ needs not hold.

We can now pose a result regarding robust Lyapunov stability of the interconnection:

Corollary 17: [Dissipativity implies robust Lyapunov stability] Take the same assumptions as in the theorem. Let $\bar{x}$ be a strict local minimum point of the storage function V and assume that V is continuous in a neighbourhood Ω of $\bar{x}$. Then there exists another neighbourhood $\Omega' \subset \Omega$ of $\bar{x}$ such that the following holds: If the initial state x_0 is in Ω', and if the positive bounds β_i, β_v are small enough, then the state $x(t)$ remains in Ω for $t \in [0, \infty)$; in addition the performance bound

$$\int_0^\infty l \; dt \leq V(x_0) - V(\bar{x}) + \sum_i d_i \beta_i + d_v \beta_v$$

holds. □

The proof of the corollary is conceptually identical to standard Lyapunov stability proofs, e.g. [59], although some extra technicalities are needed because $V(x(t))$ is not necessarily a non-increasing function of time.

Proof: Set $\alpha = V(\bar{x})$. Assume without loss of generality that Ω is bounded and that $\inf_{x \in \Omega} V(x) = \alpha$: If not, then replace Ω with $\Omega \cap B$ where B is a sufficiently small bounded neighbourhood of $\bar{x}$. Let $\gamma > \alpha$ and let $\mathbb{A}$ denote the largest connected subset of $V^{-1}([\alpha, \gamma]) \cap \Omega$ which contains $\bar{x}$; notice that $\bar{x}$ is an interior point in $\mathbb{A}$. Assume that γ is chosen such that $\mathbb{A}$ is closed; this is possible since $\bar{x}$ is a strict local minimum. Assume that β_i and β_v are small enough, i.e. $\alpha + \sum_i d_i + d_v\beta_v < \gamma$. Let Ω' be any neighbourhood of $\bar{x}$ contained in $V^{-1}([\alpha, \gamma - \sum_i d_i\beta_i - d_v\beta_v)) \cap \mathbb{A}$; Now assume that $x_0 \in \Omega'$. According to item 2 in theorem 15 $x(t)$ remains in $V^{-1}([0,\gamma])$ at least up to a finite escape time. Now apply lemma 16 to see that $x(t)$ remains in the bounded set $\mathbb{A} \subset \Omega$ for $t \in [0,\infty)$. The performance bound follows from the dissipation inequality since $V(x(t)) \geq V(\bar{x})$. ■

In the proofs above the dissipation inequality does not need to hold *everywhere* but only along the possible trajectories. This is particularly useful when studying *local* behaviour. Developments along these lines are reported in [113].

3.3 Linear systems and quadratic supply rates

In this section we specialize the previous discussion to the case of linear systems Σ defined by ordinary differential equations in state-space:

$$\Sigma: \quad \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array} \quad . \tag{3.5}$$

For systems consisting of a nominal part in feed-back with a multi-dissipative perturbation, we show that stability and various performance properties can be described by linear matrix inequalities (LMIs) which describes some dissipativity property of the nominal part. Such linear matrix inequalities can be verified directly numerically with commercially available packages such as [32, 38]. The connection between dissipativity for linear-quadratic systems and LMIs was noted already in [124] and has received much interest during the last few years [19] due to efficient numerical algorithms for solving LMI problems [82].

3.3.1 Robust stability

Consider the connection in figure 3.6 (a) where Σ is the nominal system and Δ is a perturbation in a set $\mathbf{\Delta}$; both are assumed to be causal, linear, finite dimensional, and time invariant systems. We say that the

configuration $(\Sigma, \mathbf{\Delta})$ is robustly stable if for every $\Delta \in \mathbf{\Delta}$ the configuration is well posed (i.e, the dynamics of the closed loop can be written $(\dot{x}(t), \dot{\xi}(t)) = \bar{A}(x(t), \xi(t))$ for some linear $\bar{A}$), and if furthermore $z(\cdot) \in \mathcal{L}_2$.

For a deterministic linear time invariant systems with quadratic supply rates (i.e, when $s(w,z) = (w' \;\; z')Q(w' \;\; z')'$), there is no loss of generality [124] in assuming the storage function V to be quadratic ($V(x) = x'Px$ with $P = P' \geq 0$), in which case the differential dissipation inequality (2.3) becomes [124]

$$\forall x \in \mathbb{X},\; w \in \mathbb{W} : (x' \;\; w')\Phi \begin{pmatrix} x \\ w \end{pmatrix} \leq 0$$

where Φ is shorthand for

$$\Phi = \begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q \begin{bmatrix} O & I \\ C & D \end{bmatrix}$$

This is a linear matrix inequality (LMI) in P.

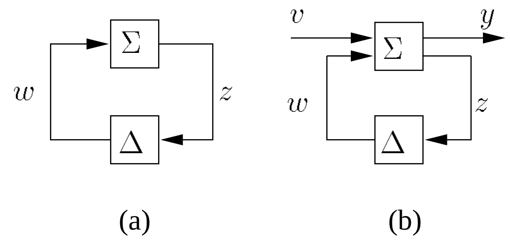

Figure 3.6: The two problems considered: (a) Robust stability. (b) Robust $\mathcal{H}_2$ performance.

Lemma 18: Assume that every $\Delta \in \mathbf{\Delta}$ is linear, time invariant, and dissipative w.r.t. $-s$, and that Σ is strictly output dissipative with respect to the supply rate s. Then the feed-back configuration $(\Sigma, \mathbf{\Delta})$ is robustly stable. □

Remark 19: If Σ is dissipative w.r.t. s but not strictly output dissipative, and if the interconnection is well posed (which in this case is not guaranteed by the dissipativity), and if the storage functions are (quadratic and) positive definite, then we have Lyapunov stability [124], but z need not be in $\mathcal{L}_2$. □

Proof: The dissipation inequalities for Σ and Δ are

$$\begin{aligned} V(x(T)) &\leq V(x(0)) + \int_0^T s(w,z) - \epsilon \|z\|^2 \, dt \quad , \\ W(\xi(T)) &\leq W(\xi(0)) + \int_0^T -s(w,z) \, dt \quad . \end{aligned}$$

Since we know that V has a local minimum (possibly non-strict) at $x = 0$, dissipativity of Σ implies that $s(w,z) - \epsilon\|z\|^2 \geq 0$ for $x = 0$. Repeating the argument for W we get that $s(w,z) \leq 0$ whenever $\xi = 0$. Combining the two we get that $z = 0$ whenever $(x,\xi) = (0,0)$. Due to linearity of the output equation of Δ we conclude that $w = 0$. This implies (using linearity) that the interconnection is well posed, and hence the solutions exist on $[0,\infty)$.

Adding the two dissipation inequalities and using the non-negativity of the storage functions give

$$\int_0^T \|z\|^2 \, dt \leq \frac{1}{\epsilon}\left(V(x(0)) + W(\xi(0))\right) \quad .$$

Since this holds for all T, the configuration is robustly stable. ■

Remark 20: It can be argued that the lemma is of limited interest since it is very restrictive to assume that perturbations are linear time invariant (a similar point was also emphasized in [106]). Notice, however, that these assumptions on Δ are only used to guarantee existence and uniqueness of the state trajectories; the $\mathcal{L}_2$-bound on z follows directly from the storage functions. Luckily, there exist other ways of guaranteeing existence and uniqueness when Δ is nonlinear and/or time varying. For instance, local well-posedness may be established through linearizations, algebraic loops may be avoided by assuming that Δ is strictly causal, and finally including w in z and assuming properness of W guards against finite escape times.

To avoid these details we will in the remainder of the chapter always assume that any feedback connection which we analyse is well-posed in the sense that there exist unique signals which solve the describing equations. We remark that if one is willing to make the assumption that Δ is LTI, then well-posedness is guaranteed by the stability conditions we derive. □

In this remainder of the chapter we consider analysis of systems Σ connected in feedback as in figure 3.6 with a perturbation Δ, which is dissipative with respect to the p supply rates

$$-s_i = -(w' \;\; z')Q_i \begin{pmatrix} w \\ z \end{pmatrix}, \qquad i \in \{1,\ldots,p\} \tag{3.6}$$

Without loss of generality we assume that Q_i are symmetric. We use the symbol $\boldsymbol{\Delta}$ to denote this particular class of *multi-dissipative* perturbations, i.e.

$$\boldsymbol{\Delta} = \{\Delta : \text{ dissipative w.r.t. } -s_i;\ i \in \{1,\ldots,p\}\} \tag{3.7}$$

To each $\Delta \in \boldsymbol{\Delta}$ and each supply rate $-s_i$ corresponds a storage function $W_i(\xi)$ defined on the state space of Δ.

Specializing theorem 15 to the linear-quadratic case leads the following result:

Theorem 21: Given the system Σ defined in (3.5) and the class of perturbations $\boldsymbol{\Delta}$ of (3.7). The configuration $(\Sigma, \boldsymbol{\Delta})$ is robustly stable if the following linear matrix inequality

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} + \epsilon \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0 \tag{3.8}$$

is satisfied for some $P = P' \geq 0,\ \epsilon > 0, d_i \geq 0$ □

We emphasize that the theorem gives a less restrictive condition than e.g. small gain criterion or positive real criterion because of the extra freedom associated with d_i.

Remark 22: One will often examine the following linear matrix inequality problem in $\bar{P}$ and $\bar{d}$ in stead:

$$\begin{bmatrix} \bar{P}A + A'\bar{P} & \bar{P}B \\ B'\bar{P} & 0 \end{bmatrix} - \sum_{i=1}^{p} \bar{d}_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} < 0 \quad ,$$

$$\bar{P} > 0,\ \bar{d}_i > 0 \quad . \tag{3.9}$$

Feasibility of this LMIP is a sufficient condition for robust stability of $(\Sigma, \boldsymbol{\Delta})$ and - under weak assumptions on the data - equivalent to feasibility of (3.8). □

Proof: Given a solution pair P, d_i, ϵ, the function $x'Px$ acts as a storage function for Σ with respect to the supply rate $\sum_i d_i s_i - \epsilon|z|^2$ and since any $\Delta \in \mathbf{\Delta}$ is dissipative w.r.t. $-\sum_i d_i s_i$ (proposition 2 on page 32 above) we have shown robust stability (lemma 18).

It is also easy to see that feasibility of (3.9) implies feasibility of (3.8): In fact, given solutions $\bar{P}, \bar{d}_i$ to (3.9), one may find sufficiently small $\epsilon > 0$ such that $\bar{P}, \bar{d}_i, \epsilon$ solves (3.8). ■

A similar result was derived independently in the recent contribution [126].

Example 23: [The conservativeness of the sufficient condition] Continuing example 13 above, the two supply rates dissipated by the perturbation Δ are $-s_1 = zw$ and $-s_2 = |z|^2 - |w|^2$ corresponding to

$$Q_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad .$$

The sufficient condition of remark 22 is that Σ is strictly output dissipative with respect to a combination of s_1 and s_2, i.e. with respect to

$$s_\alpha(w, z) = (w \ z) \begin{bmatrix} 1 & -\alpha \\ -\alpha & -1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{3.10}$$

for some $\alpha \geq 0$. Here we have taken $d_2 = 1$ which is possible due to the conicity, and $d_1 = \alpha$. This will be the case if and only if

$$(1 \ \bar{\Sigma}(s)) \begin{bmatrix} 1 & -\alpha \\ -\alpha & -1 \end{bmatrix} \begin{pmatrix} 1 \\ \Sigma(s) \end{pmatrix} > 0$$

holds for all s in the closed right half $\bar{\mathbb{C}}^+$ of the complex plane. For $\alpha \to 0$, we retrieve the condition that Σ has $\mathcal{L}_2$-gain less than 1, while for $\alpha \to \infty$ the permittable circle approaches the entire left half plane, and thus we get the condition that Σ is strictly negative real. For high order plants, the latter condition is often difficult or even impossible to obtain, while the former may impose too severe constraints on bandwidth. Also taking $\alpha \in (0, \infty)$ into account obviously increases the possibility of reaching a good design.

In comparison, the sufficient *and* necessary condition of equation (3.4) requires the existence of a function $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}_+$ such that the inequality holds. In other words, the sufficient conditions of theorem 21 and remark 22 are conservative in that they do not allow frequency dependent weights d_i.

Notice however that linearity and time invariance of Δ is essential to the derivation of equation (3.4), whereas theorem 21 holds also for nonlinear and time-varying Δ provided that the interconnection is well posed. □

Example 24: [A graphic interpretation] Continuing the preceding example, we can also give a graphic interpretation of the sufficient condition that Σ is strictly output dissipative w.r.t. s_α for some $\alpha \geq 0$: Let $S_{-\alpha}$ denote the circle in the complex plane which is centered in $-\alpha \in \mathbb{R}$ and whose boundary contains the point i - see figure 3.7. Then Σ is strictly output dissipative w.r.t. s_α if and only Σ maps the right half of the complex plane into the interior of the circle $S_{-\alpha}$. Combining with the maximum modulus theorem, the sufficient condition of remark 22 is that Σ is stable and its Nyquist plot is contained in such a circle $S_{-\alpha}$ for some $\alpha \geq 0$. This graphic criterion is reminiscent of the circle criterion for absolute stability, see e.g. [59], except that we need only find *one* suitable circle in a certain family.

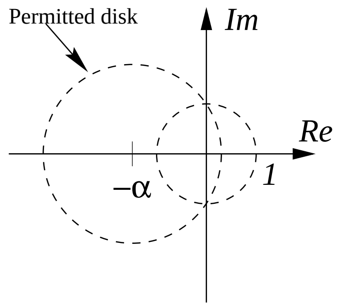

Figure 3.7: Permitted area for $\Sigma(\bar{\mathbb{C}}^+)$ with $\alpha(s)$ independent of s

A further understanding of the conservativeness of theorem 21 and remark 22 is obtained from the following observation: If Σ dissipates s_α for some $\alpha \geq 0$, then (Σ, Δ) is stable for any perturbation Δ which maps the right half of the complex plane into the circle S_α. Notice that any such circle S_α contains the original set A of figure 3.3 on page 48. The conservativeness of theorem 21 is thus illustrated by the difference between the set A and the sphere S_α which covers A. This interpretation is not restricted to this particular example, but applies to theorem 21 and remark 22 in general. □

3.3.2 Parameter uncertainty

A popular model of parameter uncertainty is that the system matrices A, B, C and D of the system (3.5) are time varying but remain in a given polytope:

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathbf{Co}\left(\left\{\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \mid j = 1, \ldots, m\right\}\right) \quad . \tag{3.11}$$

Here $\mathbf{Co}(\cdot)$ denotes convex hull. This situation covers not only parameter uncertainties but also non-linear systems with sector-bounded non-linearities. In [19] a model like this is called a *Polytopic Linear Differential Inclusion*, and many properties of such models are reduced to feasibility of LMIs. It is therefore not surprising that also robustness in the presence of multi-dissipative perturbations can be guaranteed with LMIs. For this situation, a sufficient condition for robust stability in the presence of multi-dissipative perturbations is given by the following:

Theorem 25: Let Σ satisfy (3.11). Assume that each of the supply rates s_i is concave in z and that the following LMI problem in P, d_i and ϵ is feasible:

$$\forall j \in \{1, \ldots, m\}:$$

$$\begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix}$$

$$+\epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} \begin{bmatrix} C_j' & D_j' \end{bmatrix} \leq 0 \quad ,$$

$$P \geq 0, \; d_i \geq 0, \; \epsilon \geq 0 \quad .$$

Then the feed-back configuration $(\Sigma, \mathbf{\Delta})$ is robustly stable. □

Remark 26: The assumption that s_i is concave in z may be written

$$Q_i^{zz} \leq 0 \text{ where } Q_i = \begin{bmatrix} Q_i^{ww} & Q_i^{wz} \\ Q_i^{zw} & Q_i^{zz} \end{bmatrix}$$

and essentially says, that when the input w is zero, the flow s_i is zero or directed out of the system Σ. □

Proof: The proof consists of tedious though straightforward manipulations of linear matrix inequalities and can be found in appendix B.1 on page 167. The idea is that the time-varying system is strictly output dissipative with respect to the supply rate $\sum_i d_i s_i$; as storage function we use the time-invariant function $V(x) = x'Px$. ■

3.3.3 Guaranteed $\mathcal{H}_2$ Performance

We now expand the system with an exogenous input $v(t)$ and a performance output $y(t)$, corresponding to figure 3.6 (b) on page 55:

$$\begin{aligned} \dot{x} &= Ax + Bw + Gv \\ z &= Cx + Dw \\ y &= Hx + Jw \end{aligned} \tag{3.12}$$

As before, we have $w = \Delta z$ where $\Delta \in \mathbf{\Delta}$. We use the symbol (Σ, Δ) to denote the closed-loop system with input v and output y. As a measure of performance for (Σ, Δ) we use its $\mathcal{H}_2$-norm.

When Δ is nonlinear and/or time-varying one needs to specify what is meant by the $\mathcal{H}_2$-norm of the interconnection, since it cannot be represented by a transfer function. Two possibilities exist: One can use the $\mathcal{L}_2$-norm of the impulse response which we will call the deterministic $\mathcal{H}_2$-norm of the interconnection, or one can assume that v is white noise and consider the steady-state variance of y. At this point we discuss the deterministic interpretation while the stochastic approach is taken in the second part of this thesis, in chapter 6 below.

Theorem 27: The $\mathcal{H}_2$ norm of the closed loop system from v to y is bounded above by

$$\|(\Sigma, \Delta)\|^2_{\mathcal{H}_2} \leq \inf_{P, d_i, \epsilon} \operatorname{tr}(G'PG)$$

where the infimization is subject to

$$P = P' \geq 0, \; d_i \geq 0, \; \epsilon > 0$$

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} - \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} H' \\ J' \end{bmatrix} \begin{bmatrix} H & J \end{bmatrix} + \epsilon \begin{bmatrix} C \\ D \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0$$

□

Remark 28: Computation of the upper bound on $\mathcal{H}_2$-performance is an LMI problem in P, d_i and ϵ.

Notice that if we remove Δ, the LMI in P reduces to $PA+A'P+H'H \leq 0$, i.e. we retrieve the standard way of computing the $\mathcal{H}_2$-norms of known system using the obervability Gramian [128]. □

Proof: Given P, d_i and ϵ, we know that Σ *with* $v \equiv 0!$ is strictly output dissipative w.r.t. $\sum_i d_i s_i - ||y||^2$. This implies strictly output dissipativity w.r.t. $\sum_i d_i s_i$ and hence it is reasonable to assume well-posedness of the interconnection of Σ and Δ, cf. remark 20.

Assume that the interconnection (Σ, Δ) is at rest for $t < 0$ and that we at $t = 0$ excite the interconnection with an impulse at v, i.e. $v(t) = v_0\delta(t)$ where $\delta(\cdot)$ is the Dirac delta. We then have $x(0^+) = Gv_0$. Assume that P, d_i and ϵ solve the LMI problem in the theorem, then the integral dissipation inequality for the interconnection reads

$$\int_0^T ||y||^2 \ dt + \int_0^T \epsilon ||z||^2 \ dt \leq x'(0^+)Px(0^+)$$

(and holds because $v(t) = 0$ for $t > 0$). Hence,

$$\int_0^\infty ||y||^2 \ dt \leq v_0' G' P G v_0 \quad .$$

Now let v_j be the jth unit vector in the input space $\mathbb{V} = \mathbb{R}^{n_v}$ $(v(t) \in \mathbb{V})$ and let $y_j(t)$ be the impulse response of the interconnection (Σ, Δ) to the input $v(t) = v_j\delta(t)$. We then have

$$||(\Sigma, \Delta)||^2_{\mathcal{H}_2} = \sum_{i=1}^{n_v} \int_0^\infty ||y_j(t)||^2 \ dt \leq \sum_{i=1}^{n_v} v_i' G' P G v_i = \text{trace } G'PG \quad .$$

Since this holds for any P, d_i and ϵ that solve the LMI problem the conclusion in the theorem follows. ■

3.4 Chapter conclusion

The concept of dissipation is widely used in the area of robust control and control of large scale systems, but except for the special cases of μ theory [28, 128] or more generally integral quadratic constraints [77, 57, 97], there has been no systematic use of the fact that systems possess several dissipation properties at once.

In this chapter we have reported results on the use of such multiple properties of dissipation. Our results essentially follow from the fact that the supply rates dissipated by a given system form a convex cone. We have derived results corresponding to robust Lyapunov stability as well as robust performance. The framework allows generalization of several other standard Lyapunov-type results; of particular practical relevance is ultimate boundedness, slowly varying systems and parametric uncertainty. Many such extensions are straightforward.

The appeal of the framework is that it allows combination of information and specifications of different types. Admittedly the resulting conditions will be conservative in that only sufficient conditions are given. Compared to common practice, however, where either several dissipation properties of the involved uncertain subsystems are ignored or the uncertainty is simply left out of the analysis, the framework is an improvement.

It is appealing that the analysis reduces to linear matrix inequalities in the special, but very important, case of linear systems and quadratic supply rates. For nonlinear systems the issue of numerical methods is more critical; see the note below.

3.5 Notes and references

Comparison to the IQC framework

Consider a perturbation Δ which maps $z(\cdot)$ to $w(\cdot)$ and which is dissipative with respect to a supply rate $s(z, w)$ which is quadratic, i.e. $s(z, w) = (w' \ z')Q(w' \ z')'$. Assume that the available storage is 0 at time 0; then the signals satisfy the *integral quadratic constraint* (IQC)

$$\int_0^T (w'(t) \ z'(t))Q(w'(t) \ z'(t))' \ dt \geq 0$$

for all times T. The converse also holds: If the IQC holds for all inputs $z(\cdot)$, and if the state space of the perturbation Δ is reachable, then Δ is dissipative w.r.t. s with available storage 0 at time 0.

With this perspective, it is reasonable to compare our framework of multi-dissipative perturbations to that of integral quadratic constraints. Clearly the motivation behind the two frameworks are identical as is the idea of *modularity*: reducing a large complex problem to a collection of smaller and more managable subproblems, viz. performing dissipation (or IQC) analysis on components. The techniques used are quite different, though.

The IQC framework, in the sense of [77, 57], makes heavy use of frequency-domain techniques. Although it is feasible to pose IQCs for specific nonlinear perturbations, see for instance [57], the resulting conditions on the nominal system are in frequency domain and no results are given as to how to verify these conditions for nonlinear nominal systems. The sufficient conditions are less conservative than the ones we have obtained in this chapter in that they make use of frequency dependent *multipliers* corresponding to our weights d_i (c.f. examples 13 and 23 above). In order to make use of these extra degrees of freedom in the *linear* case one needs to consider infinite-dimensional convex optimization problems associated with the choice of multipliers; this is the major numerical challenge. In comparison the hurdle in our framework of multi-dissipative perturbations is the computation of storage functions for *nonlinear* nominal systems, which also can be cast as an infinite-dimensional convex optimization problem.

Another approach to integral quadratic constraints is found in [95, 96, 97]. These papers use time-domain techniques and the approach is closer to this chapter than is [77, 57]. Only linear nominal systems are considered as are integral quadratic constraints corresponding to $\mathcal{L}_2$-gain of the perturbations.

Numerical methods for optimal control problems

In order to verify if a given system dissipates a given supply rate one needs to consider the optimal control problem which defines the available storage or the required supply. Except for systems with low dimensional state spaces or a particular structure, this is an overwhelming numerical challenge which is the major obstacle to the practical use of the results in this chapter.

Among the numerical methods for optimal control problems, those based on dynamic programming rather than the maximum principle are most natural: In fact the optimal trajectories are of less interest whereas approximations of the value functions may serve as storage functions.

For a fixed supply rate, storage functions may be approximated by discretization of the differential dissipation inequality, [52, 65] or by a spectral

method where a storage function is sought in a given finite-dimensional space [11]. An alternative is a recursive scheme due to Lukes [73, 120, 78] for computing the Taylor expansion of the value function around an isolated equilibrium point.

The sufficient conditions for robustness presented in this chapter require finding a storage function *and a supply rate* simultaneously which adds an extra twist to the optimal control problem. One approach is to restrict the storage function to a finite dimensional space and employ convex optimization techniques, optimizing over the storage function and the d-weights simultaneously. For input affine-quadratic systems, the LMI based procedure described on page 20 may easily be modified to search simultaneously for the storage function V and the set of weights d_i, d_l required by theorem 15. Although the convexity makes a convergence analysis feasible, the size of the optimization problems grows exponentially with the number of states; this is Bellman's curse of dimensionality. More heuristic approaches may be useful. For instance we presented in [113] an example where the d-weights first where fixed considering only the linearization of the system; afterwards higher order terms were included using a Lukes' scheme.

State feed-back controller design

We briefly comment on the problem of finding a state feedback controller $u(t) = \mu(x(t))$ such that the resulting closed loop system satisfies the sufficient condition derived in this chapter.

For a *fixed* supply rate, the problem of control for dissipation requires practically the same tools as the problem of dissipation analysis as is evident in [120]. This reference treats the special problem of $\mathcal{L}_2$-gain analysis and nonlinear $\mathcal{H}_\infty$ control, but the discussion applies to broad classes of supply rates: In stead of optimal control problems we consider differential games, and the differential dissipation inequality is replaced by a Hamilton-Jacobi-Isaacs equation. In both cases the Hamiltonian dynamics provides information about existence of a value function. The issue of smoothness of storage functions becomes more problematic since control strategies are found from the partial derivatives of the value functions; see [7, 105]. Local approximations to value functions may be found by Lukes' scheme, [120, 78].

To employ the sufficient conditions in this chapter, we need to find a control law, a storage function, *and a supply rate*. In the reference [113] we suggested to fix the supply rate in a first step (which considered only the linearization of the system) and then apply Lukes' scheme.

An alternative is the following value-policy iteration: In the *value step*, for a *fixed* controller, we find a supply rate and a storage function such that the differential dissipation inequality holds. This analysis problem can for instance be solved with convex optimization as outlined above. Then, in the *policy step* we fix the supply rate and the storage function and compute the *maximum dissipation* controller, i.e. the control law which at each point in state space maximizes the worst-case dissipation. This is a family of static min-max problems. Then the value step and the policy step are iterated. It is easy to show monotonicity of such an algorithm; under suitable hypothesis this implies convergence. We have in [114] given the details in such an algorithm for the case of linear systems and quadratic supply rates and demonstrated it on a numerical example.

For a linear system and a quadratic supply rate it is possible to give a convex parametrization of linear controllers (static state feedback or full order output feedback) which make the system dissipative; this trick appeared first in [15] for the state feedback problem, see also [37, 126]. This motivates a two-step iterative procedure where the first step optimizes the supply rate while the second finds a controller which makes the closed loop system dissipative w.r.t. the current supply rate. A similar procedure is suggested in the recent reference [126]; see also [125].

Regarding output feedback control of nonlinear plants, it is principle possible to combine a search over the d-weights with the information state approach [55] to differential games. The resulting problems are in general deterringly complex and with the present state of the art heuristic approaches should be more fruitful; for instance, first solving the linearized problem and then applying Lukes scheme.

Towards a nonconservative condition

The technique in this chapter is essentially the following: If $V(x)$ is a storage function for Σ with respect to $s+\sum_i d_i s_i$ and $V_a(\xi, -s_i)$ are storage functions for Δ w.r.t. $-s_i$, then $V(x)+\sum_i d_i V_a(\xi,-s_i)$ is a storage function for the interconnection (Σ, Δ) w.r.t. s. One way to generalize this is to find a function $\bar{V}(x, \beta_i)$ such that the available storage of (Σ, Δ) is less than $\bar{V}$ provided that $V_a(\xi, -s_i) \leq \beta_i$. This leads to a less conservative

condition since we do not require $\bar{V}$ to be in the form $V(x) + \sum_i d_i\beta_i$. In fact this condition is nonconservative in a certain sense, and can be verified by performing dissipation analysis of an extended plant. We do not pursue this further at this point but will return to the stochastic analogy of the idea in part II of this dissertation; see page 133.

Chapter 4

Simultaneous $\mathcal{H}_\infty$ Control

We consider the problem of finding one output feedback controller which achieves $\mathcal{H}_\infty$ performance when connected to any one of p linear time invariant plants. This is a prototype of an adaptive $\mathcal{H}_\infty$ control problem. We formulate the problem as a non-linear $\mathcal{H}_\infty$ problem and show that the minimax controller is finite dimensional but not based on certainty equivalence. Synthesis of the minimax controller involves solving a partial differential equation, namely a state feedback Hamilton-Jacobi-Isaacs equation. We investigate the structure of the solution and derive the heuristic certainty equivalence controller which has a switching architecture.

4.1 Introduction

Robustness in presence of both parametric and dynamic perturbations is an important problem which poses great theoretical difficulties. In applications, parametric uncertainty is typically effective at low frequencies, and is often highly structured. On the other hand, less structured dynamic perturbations always affect high frequency behaviour [128, p. 216].

With a low level of parametric uncertainty and with a $\mathcal{H}_\infty$ bounded dynamic perturbation, *linear* controllers may suffice, which then can be designed using μ synthesis [128, 5] or quadratic stabilization [130, 39, 15]; see also [34]. For larger levels of parametric uncertainty one would expect that improvement can be achieved by using nonlinear controllers which include an adaptation mechanism. This motivates the field of adaptive $\mathcal{H}_\infty$ control.

A natural approach to adaptive $\mathcal{H}_\infty$ control is to *extend* the state with the unknown parameters, thus obtaining a $\mathcal{H}_\infty$ control problem for a new nonlinear plant. Then, one may apply the differential game techniques [9, 120, 55] to nonlinear $\mathcal{H}_\infty$ control. This approach has been pursued in for instance [22, 25]. In these references uncertainty is restricted to special parts of the system such that the minimax controller is finite dimensional and based on certainty equivalence principles such as the one in [14].

In view of this, an immediate question is: With a dynamic game approach to adaptive $\mathcal{H}_\infty$ control, is certainty equivalence and finite-dimensional minimax controllers the generic situation, or a special case? To study this question we consider the special situation where the unknown parameter *a priori* is restricted to a known, finite set. Such problems of *simultaneous control* can be considered as a prototype of adaptive control problems - see e.g. [44]. Our conclusion is negative: Certainty equivalence can *not* be expected to hold in adaptive $\mathcal{H}_\infty$ control problems. Furthermore, the minimax controller must run a linear $\mathcal{H}_\infty$ filter for each possible value of the parameter. Therefore we expect the minimax controller to be infinite-dimensional when there is a continuum of possible parameter values.

Next, we show that the *heuristic* certainty equivalence controller guarantees $\mathcal{H}_\infty$ performance, provided that the minimax control input is uniquely defined for *almost* all times. This *weak* certainty equivalence principle emphasizes the following point: The important issue is not if the *best* (i.e, minimax) controller is based on certainty equivalence, but if a certainty equivalence based controller is *good enough,* i.e, guarantees that the $\mathcal{H}_\infty$ design objective is met.

Besides being prototypes of adaptive control problems, simultaneous control problems have been the subject of considerable independent research. Linear controllers are investigated in [17, 19]; in general nonlinear control leads to improvement. *Switching control* is studied in [79, 80, 81] and the references therein: These controllers consist of a bank of linear low-level controllers and a high-level logical switch, which connects one of the low-level controllers to the plant. One way of designing the switch is to find that *estimator* which supplies the best fit with observations, and then switch in the corresponding controller. This technique appears in [81, 79, 49].

In this chapter we point out that this technique, when applied to the problem of simultaneous $\mathcal{H}_\infty$ control, must be modified so that the switch compares not just estimation errors but also a *control error* associated with each controller. The resulting switching controller is exactly the heuristic certainty equivalence controller.

A problem with switching architectures is chattering. Chattering is rapid switching back and forth, or that unique (classical) solutions to the dynamic equations seize to exist, depending on ones point of view. Modification of the switch to avoid chattering are suggested in [79]; in this chapter we suggest an alternative based on a smooth approximation of the switch.

The chapter is organized as follows: Section 4.2 formulates the simultaneous $\mathcal{H}_\infty$ control problem. Section 4.3 deals with the extended state feedback problem while section 4.4 develops the filter for the worst-case extended state estimate. Section 4.5 discusses the the minimax controller. Section 4.6 concerns the heuristic certainty equivalence controller. Finally section 4.7 offers some conclusions.

Notation

If P is a two-port plant with disturbance input w, control input u, measurements y and error signal z, and K is a controller with input y and output u, then (P, K) denotes the closed-loop system with inputs w and outputs z (see figure 4.1 below).

We use the standard notion of $\mathcal{L}_2$-gains, see [119] and page 17 above:

Definition 29: [$\mathcal{L}_2$-gain] The $\mathcal{L}_2$-gain of a state-space system Σ (mapping inputs $w(\cdot)$ to outputs $z(\cdot)$ through states $\zeta(\cdot)$) is denoted $||\Sigma||$ and is the infimum of all numbers $\gamma > 0$ such that

$$\forall \zeta_0 : \ \exists M(\zeta_0) : \ \forall t_f > t_0, \ w \in \mathcal{L}_2([t_0, t_f]) :$$

$$\int_{t_0}^{t_f} |z(t)|^2 \, dt \le \gamma^2 \int_{t_0}^{t_f} |w(t)|^2 \, dt + M(\zeta_0) \quad .$$

Here $z(\cdot)$ is the output corresponding to the input $w(\cdot)$ and the initial state $\zeta(t_0) = \zeta_0$. If no such γ exists we write $||\Sigma|| = \infty$. □

We consider only measurable locally bounded inputs and assume that all systems map such inputs to measurable locally bounded states and outputs.

4.2 Problem statement

We consider systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & A_\theta x(t) + B_\theta u(t) + G_\theta w(t) \\ y(t) & = & C_\theta x(t) + v(t) \\ z(t) & = & \begin{pmatrix} H_\theta x(t) \\ u(t) \end{pmatrix} \\ \theta & \in & \{1, \ldots, p\} \stackrel{\triangle}{=} \Theta \end{array} \tag{4.1}$$

Here, x is the state of the system, u is a control input, w is an process disturbance, y is the measured signal, v is the measurement noise, z is the generalized error signal. All signals take values in Euclidean spaces.

The matrices $(A_\theta, B_\theta, G_\theta, C_\theta, H_\theta)$ are known functions of the unknown parameter θ. With P_θ we denote the linear system from (w, v, u) to (z, y) obtained by fixing θ.

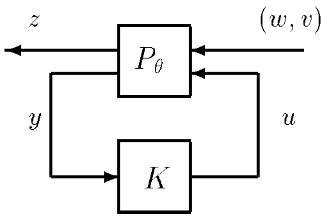

Figure 4.1: Simultaneous Control Problem

Problem of Simultaneous $\mathcal{H}_\infty$ Control with Stability: Given a constant $\gamma > 0$, find a causal control law $K: y(\cdot) \to u(\cdot)$ such that for any parameter $\theta \in \Theta$, the closed-loop system (P_θ, K) from (w, v) to z has $\mathcal{L}_2$-gain less than γ and in addition (P_θ, K) is internally stable in the sense that $w(\cdot) \in \mathcal{L}_2([0,\infty))$, $v \in \mathcal{L}_2([0,\infty))$ implies that $x(t) \to 0$ as $t \to \infty$. $\diamond$

We adopt the following standard assumptions on the system matrices:

Assumption 30:

1. For any $i \in \Theta$, the triple (C_i, A_i, B_i) is detectable and stabilizable.
2. For any $i \in \Theta$, the triple (H_i, A_i, G_i) is observable and controllable.

$\square$

The standard discussion regarding these assumptions applies [128]: The first part is necessary for the existence of an internally stabilizing controller. The second part is mainly a technical regularity assumption, which guarantees that any closed-loop system with finite $\mathcal{L}_2$-gain must be internally stable and furthermore that certain loss functions are positive definite. As in linear $\mathcal{H}_\infty$ control for a fixed plant, the second assumption can be relaxed quite a bit but to keep the exposition simple, we shall not do this.

Adding the parameter dynamics

$$\dot{\theta}(t) = 0 \tag{4.2}$$

we obtain a non-linear system description in the *extended state* $(x', \theta)'$ by combining (4.2) and (4.1). We then attack the problem of non-linear $\mathcal{H}_\infty$ control for this system using the *differential game* techniques for output feedback design presented in [9, 120]. To be specific, we consider the differential min-max problem

$$\min_K \sup_{w(\cdot), x(\cdot), \theta} \left[\int_0^\infty -s_\theta(t)\ dt - \frac{1}{2} x(0)' N_\theta x(0) - \Lambda_\theta \right] \tag{4.3}$$

where we have used the shorthand

$$s_i = \frac{1}{2}\gamma^2 |w|^2 + \frac{1}{2}\gamma^2 |y - C_i x|^2 - \frac{1}{2}|u|^2 - \frac{1}{2}|H_i x|^2 \quad . \tag{4.4}$$

Here, the supremum is subject to the dynamics (4.2) and (4.1), and the minimization is subject to the causality restriction on the controller K. Notice that the initial condition $x(0)$ is chosen by the maximizing player. If this minimax problem has finite upper value for some choice of N_i and Λ_i, then there exists a controller (viz., the minimax controller) which guarantees that the closed loop from $(w(\cdot), v(\cdot))$ to $z(\cdot)$ has $\mathcal{L}_2$-gain less than or equal to γ. The converse also holds.

In this minimax problem $\Lambda_i \geq 0$ represents prior information about θ; our prior estimate of θ is $\arg\min_i \Lambda_i$ (assuming a unique minimum point). For simplicity and without loss of generality we assume $\min_i \Lambda_i = 0$.

Similarly, $N_i > 0$ represents our confidence in the prior estimate $x(0) = 0$, given that $\theta = i$. The choice of N_i influences the transients of the state estimator but not steady-state behaviour such as the closed-loop $\mathcal{L}_2$-gain. The standard discussion from linear $\mathcal{H}_\infty$ theory [42] applies; the situation corresponds to the initialization of variances in Kalman filters.

For simplicity, we are going to assume that the estimator starts in steady state; see section 4.4 below.

In the following sections we approach this min-max problems, following the general procedure of [9] closely as far as possible. It is interesting to see that no problems are caused by the fact that our state space $\mathbb{R}^n \times \{1,\dots,p\}$ is *hybrid*, i.e. has a continuous as well as a discrete part. The main steps in the procedure are:

1. The full information problem where x and θ is available to the controller on-line. This problem reduces to p standard linear $\mathcal{H}_\infty$ problems; section 4.3.
2. The problem of estimating x and θ using the measured signal $y(\cdot)$. The solution is a *bank* of linear state estimators, one for each parameter value, which run in parallel. The final state estimate is found by comparing residuals associated with these estimators; section 4.4.
3. In [9], a certainty equivalence principle [14] is verified at this point. In our case, the hypothesis for this principle is not met. In stead, we reduce the problem to a finite-dimensional full information minimax control problem. Our procedure is similar to the information state machinery [55]. The minimax controller is then characterized by a Hamilton-Jacobi-Isaacs equation. We discuss this equation and the structure of its solution; section 4.5.
4. Finally we investigate the *heuristic* certainty equivalence controller; section 4.6.

4.3 Control with known extended state

We address the subproblem where $y = (x,\theta)$. A trivial but helpful observation is that this *extended state feedback* problem reduces to a standard *linear* $\mathcal{H}_\infty$ problem *for each parameter* θ. Following [128], we consider the p control algebraic Riccati equations

$$A_i'X_i + X_iA_i + X_i\left(\frac{1}{\gamma^2}G_iG_i' - B_iB_i'\right)X_i + H_i'H_i = 0 \qquad (4.5)$$

which we explicitly assume have the needed solutions:

Assumption 31: For each $i = 1,\dots,p$, the algebraic Riccati equation (4.5) admits a solution X_i such that $A_i - B_iB_i'X_i + \frac{1}{\gamma^2}G_iG_i'X_i$ is asymptotically stable. In addition X_i is positive semi-definite. □

Remark 32: For the relevant theory of Riccati equations as (4.5) we refer to [128]. We note if such an X_i exists, it must be unique. Furthermore X_i must be positive definite since (H_i, A_i) is assumed observable. □

Well known results from linear $\mathcal{H}_\infty$ theory thus immediately gives:

Proposition 33: [c.f. [128, theorem 16.9], [9, theorem 4.8]] Let the plant (4.1), (4.2) satisfy assumption 30. Then there exists a causal control law $(\theta(\cdot), x(\cdot)) \to u(\cdot)$ such that the closed-loop system from $w(\cdot)$ to $z(\cdot)$ is internally stable and has $\mathcal{L}_2$-gain less than γ, if and only if assumption 31 holds. In this case, one such control law is the minimax control

$$u(t) = -B_\theta' X_\theta x(t) \quad . \qquad (4.6)$$

The associated cost-to-go is

$$P(x_t,\theta) \triangleq \sup_{w(\cdot)} \int_t^\infty \frac{1}{2}|z(\tau)|^2 - \frac{1}{2}\gamma^2|w(\tau)|^2 \, d\tau = \frac{1}{2}x_t'X_\theta x_t \qquad (4.7)$$

where the supremum is subject to the initial condition $x(t) = x_t$ and the dynamic equations (4.1,4.2,4.6) governing the closed loop. △

4.4 The estimation problem

In this section we define the problem of estimating the extended state and derive the dynamic filter of the estimator. As in [9, 25, 120], we define the *cost-to-come* function (termed the *information state* by other authors, e.g. [55])

$$R(x_t,i,t) = \inf_{w(\cdot),x(\cdot)} \left(\int_0^t s_i(\tau)\, d\tau + \frac{1}{2}x'(0)N_ix(0)\right) + \Lambda_i \qquad (4.8)$$

where s_i is as in (4.4). The infimization in (4.8) is subject to the constraints

$$\begin{aligned} x(t) &= x_t \quad , \\ \dot{x}(\tau) &= A_ix(\tau) + B_iu(\tau) + G_iw(\tau) \ , \quad 0 \le \tau \le t \ . \end{aligned}$$

The cost-to-go is the worst-case loss over the time interval $[0,t]$, given $y(\cdot)$ and $u(\cdot)$ and assuming that $x(t)=x_t$ and that $\theta=i$.

Following the notation in [120], we denote by $S(x,i,t)$ the worst-case total cost over the time interval $[0,\infty)$ consistent with the observations of $u(\tau), y(\tau)$ for $\tau\in[0,t]$ and such that $x(t)=x$, $\theta=i$, and subject to full information control for $\tau\geq t$. Hence

$$S(x,i,t)=R(x,i,t)-P(x,i) \quad .$$

We can now define the worst-case extended state estimate:

$$\begin{pmatrix} \hat{x}(t) \\ \hat{\theta}(t) \end{pmatrix} = \arg\min_{x,i} S(x,i,t) \quad . \tag{4.9}$$

The extended state estimate is instrumental to the minimax controller: A certainty equivalence controller [14, 9, 120] applies the full information control law (4.6) with the state x,θ substituted with $\hat{x},\hat{\theta}$. Without certainty equivalence, we demonstrate in the following section that the problem can be transformed into one where the extended state estimate $\hat{x},\hat{\theta}$ is the controlled variable.

In order to derive the dynamics of the extended state estimate we split the estimation into two parts: First a conditional state estimate which estimates x conditioned on assumptions on θ, and second the (unconditional) parameter estimate. To be specific, the conditional state estimate is

$$\xi(i,t)= \arg\min_{x} S(x,i,t) \tag{4.10}$$

and is the worst-case state estimate based on the assumption that the true parameter equals i. Correspondingly the worst case parameter estimate is

$$\hat{\theta}(t)= \arg\min_{i} S(\xi(i,t),i,t) \quad . \tag{4.11}$$

With this formulation the state estimate is $\hat{x}(t)=\xi(\hat{\theta}(t),t)$. Determining $\xi(i,t)$ for fixed i is a purely linear problem which can be solved as in [9, 128]:

Assumption 34: For each $i=1,\ldots,p$, the filter algebraic Riccati equation

$$Y_iA_i'+A_iY_i+G_iG_i'+Y_i(\frac{1}{\gamma^2}H_i'H_i-C_i'C_i)Y_i=0 \tag{4.12}$$

admits a positive semi-definite solution Y_i such that $A_i'+(\frac{1}{\gamma^2}H_i'H_i-C_i'C_i)Y_i$ is asymptotically stable. □

By duality of remark 32, such a Y_i will be unique and positive definite. Define $Q_i:=\gamma^2Y_i^{-1}$. then Q_i satisfies

$$A_i'Q_i+Q_iA_i+\frac{1}{\gamma^2}Q_iG_iG_i'Q_i+H_i'H_i-\gamma^2C_i'C_i=0 \quad .$$

For ease of notation we assume that the game (4.3) has been chosen such that $Q_i=N_i$ for all $i=1,\ldots,p$; thus the filters start in steady-state. See the discussion on page 74 above, and appendix B.2.

The implication of assumption 34 is that the cost-to-go is always well defined and for each i has a minimum over x which is attained at a unique point. For the same to hold for $S(x,i,t)$ we need $S(x,i,t)$ to be strictly convex in x, i.e:

Assumption 35: For each $i=1,\ldots,p$, the coupling condition

$$Q_i-X_i>0$$

holds. □

Summarizing, linear $\mathcal{H}_\infty$ theory gives us the following proposition:

Proposition 36: Let the plant (4.1), (4.2) satisfy assumption 30. There exist causal controllers $K_i: y(\cdot)\to u(\cdot)$ such that (P_i,K_i) are internally stable and have $\mathcal{L}_2$-gain less than γ if and only if assumptions 31, 34 and 35 hold. Assume in addition that $N_i=Q_i$, then $\xi(i,t)$ is well defined for all t and all $u\in\mathcal{L}_2([0,t])$, $y\in\mathcal{L}_2([0,t])$ and can be computed on-line as the solution to the ODE

$$\begin{aligned}\dot{\xi}(i,t) = \; & \\ & \left(A_i+\gamma^{-2}G_iG_i'X_i-B_iB_i'X_i\right)\cdot\xi(i,t) \\ & +\gamma^2(Q_i-X_i)^{-1}C_i'\cdot(y(t)-C_i\xi(i,t)) \\ & +(Q_i-X_i)^{-1}Q_iB_i\cdot(u(t)+B_i'X_i\xi(i,t))\end{aligned} \tag{4.13}$$

with initial condition $\xi(i,0)=0$. Furthermore the conditional worst-case loss $S(\xi(i,t),i,t)$ is computed on-line as the solution to the ODE

$$\frac{d}{dt}S(\xi(i,t),i,t)=\frac{1}{2}\gamma^2|y(t)-C_i\xi(i,t)|^2-\frac{1}{2}|u(t)+B_i'X_i\xi(i,t)|^2 \tag{4.14}$$

with the initial condition $S(\xi(i,0),i,0)=\Lambda_i$. △

All statements in the proposition can be found in [9, theorem 5.5] (see also [128, theorem 16.4]) *except* the dynamic equations for ξ and S. Nevertheless, these equations can easily be derived using the method of [9, 120]; the calculations can be found in appendix B.2.

The structure of the single estimator $\xi(i,\cdot)$ is illustrated in figure 4.2 where we have omitted the subscript i and used the notation

$$\begin{aligned} E &:= \gamma^{-2}G'X \\ F &:= -B'X \\ K &:= (Q-X)^{-1}QB \\ L &:= \gamma^2(Q-X)^{-1}C' \end{aligned}$$

The block diagram (and the ODEs) for the conditioned state estimate $\xi(i,t)$ is identical to estimator in the standard central $\mathcal{H}_\infty$ controller [128, p. 435], *except* for the last term $(Q_i - X_i)^{-1}Q_iB_i\,(u(t) + B_i'X_i\xi(i,t))$ (the block K in the block diagram). This term vanishes when the control signal is conditionally minimax (i.e, $\tilde{u} = 0$ as will happen when $\hat{\theta}(t) = i$ and certainty equivalence control is used; see below) and is therefore not present in the central $\mathcal{H}_\infty$ controller for a single linear plant. The way $\tilde{u}$ affects the dynamics of the conditional state estimate corresponds to a parametrization of all $\mathcal{H}_\infty$ suboptimal controllers [128, p. 420] (we will elaborate further on this connection in remark 38 below).

We see from equation (4.14) that $S(\xi(i,t),i,t)$ is an integrated *residual* associated with the model P_i. The *estimation error* $y(t) - C_i\xi(i,t)$ appears also in residuals of stochastic system identification, but the subtraction of the *control error* $u(t) + B_i'X_i\xi(i,t)$ is a new feature due to the minimax setting. Notice that $-B_iX_i\xi(i,t)$ is an *estimate* of the full information minimax control (4.6).

In the remainder of the chapter we will use the shorthands

$$\xi_i(t) := \xi(i,t) \quad \text{and} \quad S_i(t) := S(\xi(i,t),i,t) \quad .$$

The total cost function $S(x,i,t)$ can be computed as

$$S(x,i,t) = \frac{1}{2}(x - \xi_i(t))'(Q_i - X_i)(x - \xi_i(t)) + S_i(t)$$

after which the cost-to-come function can be computed as

$$R(x,i,t) = S(x,i,t) + P(x,i) \quad .$$

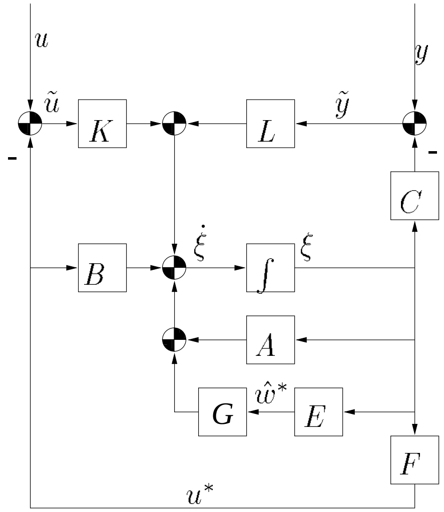

Figure 4.2: Block diagram of each conditional worst-case state estimator. The subscripts i are omitted.

4.5 The minimax controller

Having derived the minimax estimator in the previous section the first thing to verify is if the certainty equivalence (CE) principle of [14] can be applied as in [9, 120]. This principle states that *if* the worst-case extended state estimate $(\hat{x}(t), \hat{\theta}(t))$ is always well defined by equation (4.9) on page 76 in the sense that the minimum exists and is attained at a unique point, then the minimax control strategy associated with the game (4.3) on page 73 is

$$u(t) = -B'_{\hat{\theta}(t)}X_{\hat{\theta}(t)}\hat{x}(t) \quad . \tag{4.15}$$

This is a *certainty equivalence controller* since it applies the state feedback law (4.6) to the estimates $\hat{x}, \hat{\theta}$. In general, a CE principle is one which states that a CE controller is optimal (in this case minimax). If a CE controller is applied without a justifying CE principle, then we emphasize this by calling it a *heuristic* certainty equivalence controller.

We know from proposition 36 that the conditional state estimates $\xi(i,t)$ are always well defined by equation (4.10), which implies that the minimax controller in the case of a *single* plant is based on certainty equivalence [9, theorem 5.3]. However, the parameter estimate $\hat{\theta}(t)$ needs not always be well defined by equation (4.11) since the minimum of $S_i(t)$ over i may be attained for two or more values of i. In fact, if $\hat{\theta}(t)$ is always well defined then $\hat{\theta}(t)$ is a constant function of time t; clearly such an assumption would be rather detrimental to the whole idea of adaptation. We conclude that certainty equivalence does not necessarily hold.

Despite this it is possible to characterize the minimax controller implicitly in terms of a Hamilton-Jacobi-Isaacs equation; this is the subject of the remainder of this section. First we reduce the problem to a dynamic game with full information. Next we derive the Hamilton-Jacobi-Isaacs equation associated with this full information game. Finally we state a theorem and pose a conjecture about the structure of the value function.

Reduction to a full information game

At this point we adopt a technique similar to the information state machinery in order to reduce the output feedback problem to a full information game. See [55] for the information state machinery in the context of nonlinear $\mathcal{H}_\infty$ control. The corresponding approach to optimal control of Markov chains is described in [16, ch. 4].

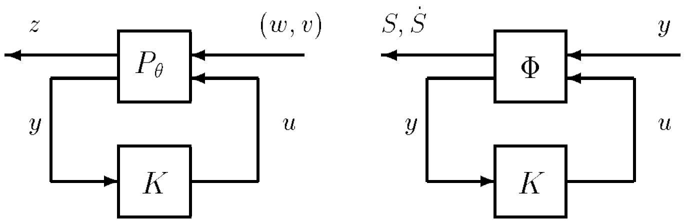

Figure 4.3: The original output feedback problem and the equivalent full information problem of controlling the total cost.

Use the symbol Φ to denote the filter with states

$$(\xi_1(t), \ldots, \xi_p(t), S_1(t), \ldots, S_p(t)) \quad ,$$

inputs u and y, outputs y, S_i and $\dot{S}_i$, and dynamics given by equations (4.13) and (4.14) above. Let $\xi = (\xi_1, \ldots, \xi_p)$ and $S = (S_1, \ldots, S_p)$. Use the symbol (Φ, K) to denote the interconnection of the filter Φ and a controller $K : y(\cdot) \to u(\cdot)$:

$$(\Phi, K)(y(\cdot)) = \Phi(K(y(\cdot)), y(\cdot))$$

The interconnection (Φ, K) thus has input $y(\cdot)$ and outputs $S(\cdot)$ and $\dot{S}(\cdot)$. See figure 4.3.

The important step in the reduction of the problem to one of full information is the following proposition, which says that as control object we may take the filter Φ rather than the plant P_θ:

Proposition 37: Let K be a causal controller $y(\cdot) \to u(\cdot)$ with a state space representation. Then the closed loop (P_i, K) has $\mathcal{L}_2$-gain less than or equal to γ *if and only if* the interconnection (Φ, K) dissipates the supply rate $\dot{S}_i$. △

The proposition follows directly from the definition of the worst-case loss S_i: (Φ, K) dissipates $\dot{S}_i$ iff $S_i(\cdot)$ can be bounded below in terms of the initial condition, and such a bound is exactly what is needed according to the definition of the $\mathcal{L}_2$ gain.

The problem of controlling Φ is essentially a *full information* problem since the initial conditions in Φ are known and all inputs to Φ are available on-line. So also the states of Φ can be considered known to the controller.

Remark 38: Loosely said, (Φ, K) dissipates $\dot{S}_i$ if and only if $u + B_i' X_i \xi_i$ is smaller than $\gamma(y - C_i \xi_i)$ in $\mathcal{L}_2$ norm. Therefore, we can construct such a controller K in the following way: Take a system $\tilde{Q}$ with $\mathcal{L}_2$-gain less than or equal to γ. Let the input to $\tilde{Q}$ be $\tilde{y} = y - C_i \xi$ and denote the output $\tilde{u}$. Now choose the control signal u such that $\tilde{u} = u + B_i' X_i \xi_i$. Thus we have established the connection to the parametrization of $\mathcal{H}_\infty$ suboptimal controllers [128, theorem 16.5], see also [9, corollary 5.2]. □

Recall that a simultaneous $\mathcal{H}_\infty$ controller was required also to be stabilizing. However, under the observability assumption 30 on page 72, any γ-suboptimal $\mathcal{H}_\infty$ controller is internally stabilizing:

Proposition 39: If (P_θ, K) has $\mathcal{L}_2$-gain less than or equal to $\gamma > 0$ and $w(\cdot) \in \mathcal{L}_2([0, \infty))$, $v(\cdot) \in \mathcal{L}_2([0, \infty))$, then $x(t) \to 0$ as $t \to \infty$. △

If only *linear* controllers K are considered, then the proposition is contained in corollary 16.3 in [128, p. 418]. A statement which allows smooth static nonlinear control is [120, prop. 3.4]; however smoothness of the storage function is required there. Given these results, there is little novelty in the proposition. For the sake of completeness we give an elementary proof.

Proof: Let L be such that $A_\theta + LH_\theta$ is asymptotically stable; such an L exists since (H_i, A_i) is assumed observable. Now write the state dynamics $\dot{x} = (A_\theta + LH_\theta)x - LH_\theta x + B_\theta u + G_\theta w$. The hypothesis implies $H_\theta x(\cdot) \in \mathcal{L}_2$ and $u(\cdot) \in \mathcal{L}_2$, so x is the state of a stable system with $\mathcal{L}_2$ inputs; thus $x(\cdot) \in \mathcal{L}_2$. Now $\dot{x}$ is a linear combination of $\mathcal{L}_2$ signals; hence also $\dot{x}(\cdot) \in \mathcal{L}_2$. Finally $x(\cdot) \in \mathcal{L}_2$ and $\dot{x}(\cdot) \in \mathcal{L}_2$ implies that $x(t) \to 0$ as $t \to \infty$. ■

Notice that the proof merely uses that $\mathcal{L}_2$ inputs w, v, are mapped to an $\mathcal{L}_2$ output z (which is somewhat weaker than finite $\mathcal{L}_2$-gain), and that the causality of K is not used.

If we ignore the slight difference between obtaining an $\mathcal{L}_2$-gain from w, v to z less than γ, and less than or equal to γ, the propositions allow us to consider a problem of *control for multi-dissipation* rather than the original problem of simultaneous $\mathcal{H}_\infty$ control. The control object in this new problem is the worst case filter Φ with states ξ, S, the control objective is to make the interconnection (Φ, K) dissipate S_i for $i = 1, \ldots, p$, and the controller has access to both the state ξ, S and the disturbance y. In summary, we have reduced the problem of simultaneous $\mathcal{H}_\infty$ control to a multi-objective min-max control problem with full information.

This multi-objective problem may be reduced to a single-objective one; in fact the interconnection (Φ, K) dissipates $\dot{S}_i$ for all $i = 1, \ldots, p$ if and only if it dissipates the supply rate

$$\frac{d}{dt} \min_i S_i(t) \quad .$$

This supply rate is regular (i.e, for any ξ and u there exists a y such that $d/dt \ \min_i S_i(t) \le 0$) and hence the problem of control for dissipation is equivalent to a differential game on infinite horizon:

$$U(\xi(0), S(0)) = \sup_{u(\cdot)} \inf_{y(\cdot)} \min_i S_i(\infty) \tag{4.16}$$

for which we can *a priori* pose the bound

$$U(\xi(0), S(0)) \in [-\infty, \min_i S_i(0)] \quad .$$

Note that the value $-\infty$ is included; the value functions in this section are in general extended real-valued. The players are allowed to play closed loop strategies which result in locally bounded $\mathcal{L}_2$ signals $u(\cdot)$, $y(\cdot)$. Note also that the limits $S_i(\infty) = \lim_{t\to\infty} S_i(t)$ are well defined for all such strategies since the filters (4.13) are stable.

The following theorem, which follows immediately from the discussion, makes precise the statement that the problem of simultaneous $\mathcal{H}_\infty$ control is equivalent to a full information game:

Theorem 40: The following are equivalent:

1. There exists a controller K such that (P_θ, K) has $\mathcal{L}_2$-gain less than or equal to γ for any θ, and such that $x(\cdot) \to 0$ for $w(\cdot), v(\cdot) \in \mathcal{L}_2$.
2. There exists a controller K such that (Φ, K) dissipates $d/dt \ \min_i S_i(t)$.
3. For each pair of initial conditions ξ, S, the lower value $U(\xi, S)$ as defined above is finite.

□

The Hamilton-Jacobi-Isaacs equation

To study the game associated with U we follow the terminology of [120] and define the *pre-Hamiltonian*

$$K(\xi, S, \lambda, \mu, u, y) = \lambda\dot{\xi} + \mu\dot{S} \tag{4.17}$$

where $\dot{\xi}$ and $\dot{S}$ are given by the filter dynamics (4.13) and (4.14) of Φ. Here λ and μ are co-states to ξ and S, respectively, i.e. λ is a row vector in $\mathbb{R}^{p\times n}$ while μ is a row vector in $\mathbb{R}^p$.

We also define the *Hamiltonian* H

$$H(\xi, S, \lambda, \mu) = \sup_u \inf_y K(\xi, S, \lambda, \mu, u, y) \quad . \tag{4.18}$$

We restrict attention to co-states for which $\sum_i \mu_i > 0$. Thus the Hamiltonian is finite, smooth, independent of S and quadratic in ξ, λ for fixed μ. Furthermore, the static game in (4.18) can be solved by completion of the squares:

$$K(\xi, S, \lambda, \mu, u, y) = H(\xi, S, \lambda, \mu) - \frac{1}{2}\sum_i \mu_i |u - u^*|^2 + \frac{1}{2}\sum_i \mu_i |y - y^*|^2$$

so Isaacs' condition[1] holds. Here y^* and u^* are smooth functions of ξ, λ and μ:

$$u^*(\xi, \lambda, \mu) = \arg\max_u \min_y K(\xi, S, \lambda, \mu, u, y) \quad ,$$

$$y^*(\xi, \lambda, \mu) = \arg\min_y \max_u K(\xi, S, \lambda, \mu, u, y) \quad .$$

The functions u^* and y^* are linear in ξ, λ for fixed μ. It is straightforward but unnecessary to give explicit expressions for these functions.

It is well known [10] that value functions of differential games, such as U, are related to Hamilton-Jacobi-Isaacs equations, in this case

$$H(\xi, S, \psi_\xi(\xi, S), \psi_S(\xi, S)) = 0 \quad . \tag{4.19}$$

The results of [10] does not cover the particular games in our study, but with analogous arguments we may obtain similar results. First we show that if (4.19) admits a *subsolution*, then it provides a *guaranteed cost strategy* for u:

Proposition 41: Let $\psi(\xi, S)$ be C^1 and satisfy $H(\xi, S, \psi_\xi, \psi_S) \geq 0$ as well as $\psi(\xi, S) \leq \min_i S_i$. Let the *maximum dissipation* controller K_ψ be specified by the state feedback law

$$u^\psi(\xi, S) = u^*(\xi, \psi_\xi, \psi_S) \quad ,$$

then (Φ, K_ψ) dissipates $d/dt \min_i S_i(t)$. Furthermore ψ is a lower bound on the lower value function: $\psi \leq U$. △

Proof: We claim that $\min_i S_i - \psi(\xi, S)$ is a storage function, i.e. that the dissipation inequality

$$\min_i S_i(T) - \psi(\xi(T), S(T)) \leq \min_i S_i(0) - \psi(\xi(0), S(0)) + \int_0^T \frac{d}{dt} \min_i S_i(t)\, dt$$

holds. This is equivalent to $\psi(\xi(T), S(T)) \geq \psi(\xi(0), S(0))$ which follows from

$$\dot{\psi} = K(\xi, S, \psi_\xi, \psi_S, u^\psi, y) \geq H(\xi, S, \psi_\xi, \psi_S) \geq 0 \quad .$$

Thus dissipation is established. Furthermore, we have

$$\min_i S_i(T) \geq \psi(\xi(T), S(T)) \geq \psi(\xi(0), S(0))$$

[1] I.e. the game in (4.18) has saddle point u^*, y^* for each ξ, S, λ, μ; see [10, p. 349].

which holds for all inputs $y(\cdot)$ and all T - hence in particular in the limit $T \to \infty$ - and thus implies that $U(\xi(0), S(0)) \geq \psi(\xi(0), S(0))$. ■

Remark 42: In the case of a single plant, $p = 1$, we may take $\psi = \min_i S_i = S_1$. The corresponding storage function is identically 0, and the resulting maximum dissipation controller is $u = -B_i' X_i \xi_i$. Thus we recover the central controller from linear $\mathcal{H}_\infty$ theory [128, p. 419]. □

The next question is if the lower value function U, or cost bounding functions corresponding to guaranteed cost strategies, must necessarily satisfy the Hamilton-Jacobi-Isaacs equation (4.19), or the related inequality. Here matters are complicated by the observation that it is not reasonable to expect U to be differentiable everywhere. Within the last decade, the notion of *viscosity* solutions [23, 35] to equations such as (4.19) has become the standard tool with which to approach these issues of non-differentiability. The following definition is taken from [23] and specialized[2] to the case of first order partial differential equations:

Definition 43: We say that $\kappa(\xi, S)$ is a *viscosity supersolution* to the Hamilton-Jacobi-Isaacs equation $H(\xi, S, \kappa_\xi, \kappa_S) = 0$ if κ is lower semi-continuous and $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) \leq 0$ holds for every $\bar{\xi}$, $\bar{S}$ and every $\phi(\xi, S)$ which is C^∞ and satisfies $\phi \leq \kappa$, $\phi(\bar{\xi}, \bar{S}) = \kappa(\bar{\xi}, \bar{S})$.

We say that κ is a *viscosity subsolution* if κ is upper semi-continuous and $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) \geq 0$ holds for every $\bar{\xi}$, $\bar{S}$ and every $\phi(\xi, S)$ which is C^∞ and satisfies $\phi \geq \kappa$, $\phi(\bar{\xi}, \bar{S}) = \kappa(\bar{\xi}, \bar{S})$.

We say that κ is a *viscosity solution* if it is both a subsolution and a supersolution. □

If κ is a viscosity supersolution, then we also say that κ solves the inequality $H(\xi, S, \kappa_\xi, \kappa_S) \leq 0$ in the viscosity sense. Notice that viscosity solutions are by definition continuous, and that a differentiable function κ is a viscosity solution if and only if it is a classical solution. We refer to [23] for further discussion of viscosity solutions.

It is by now a standard exercise to show that value functions satisfy Hamilton-Jacobi-Isaacs equations in the viscosity sense. It complicates matters, however, that the inputs u, y are not restricted to bounded sets. See page 116 for an example where the value function does not solve the PDE since near-optimal controls are unbounded. Most contributions, e.g. [70], consider only

[2] To see that our definition coincides with that in [23], substitute $F = -H$.

problems where the controls are restricted to compact sets. The recent reference [7] explicitly assumes that near-optimal controls are bounded before proving that value functions are viscosity solutions, but does not discuss how to verify the assumption for a given system.

For our system, we are able to show that the value function is indeed a viscosity solution. The key element is, roughly speaking, that controls leading to fast trajectories also lead to large running costs, as will be made precise in the proof:

Proposition 44: Assume that U is finite everywhere and continuous. Then U solves the Hamilton-Jacobi-Isaacs equation

$$H(\xi, S, \phi_\xi, \phi_S) = 0$$

in the viscosity sense. △

Proof: We show that U is a subsolution only; the other statement follows similarly. Let $\bar{\xi}$ and $\bar{S}$ be a fixed initial condition and let ϕ be a C^∞ function such that $\phi(\bar{\xi}, \bar{S}) = U(\bar{\xi}, \bar{S})$ and $\phi \geq U$. Notice that this implies that $\sum_i \phi_{S_i}(\bar{\xi}, \bar{S}) = 1$. Hence $H(\xi, S, \phi_\xi, \phi_S)$ is finite and smooth on a neighbourhood of $(\bar{\xi}, \bar{S})$.

Our proof is by contradiction: Assume that $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) < 0$. Then there exists a neighbourhood Ω of $\bar{\xi}$, $\bar{S}$ and $\delta, \epsilon > 0$ such that $\sum_i \phi_{S_i} > 2\delta$ and $H(\xi, S, \phi_\xi, \phi_S) < -\epsilon$ on Ω.

Now let $T > 0$ be arbitrary, let $\bar{T}_\Omega$ the time of first exit time from Ω, and let $T_\Omega = \min\{T, \bar{T}_\Omega\}$.

Let the minimizing player use the smooth feedback strategy $y = y^*(\xi, \phi_\xi, \phi_S)$. Let $\rho > 0$ be such that the ρ-ball around $\bar{\xi}$, $\bar{S}$ is contained in Ω. Let $c > 0$ be such that

$$\epsilon + \delta |u - u^*|^2 > c|(\dot{\xi}, \dot{S})|$$

holds for all ξ, S in Ω and all u. Such a c exists since Ω is bounded and since $\dot{\xi}$, $\dot{S}$, are affine-quadratic in u. This inequality makes precise the statement that controls leading to fast trajectories also lead to large running costs.

Thus, for any strategy for the maximizing player, we have

$$\begin{aligned}
\phi(\xi(T_\Omega), S(T_\Omega)) - \phi(\bar{\xi}, \bar{S}) &= \int_0^{T_\Omega} K(\xi, S, \phi_\xi, \phi_S, u, y^*)\ dt \\
&\leq -\int_0^{T_\Omega} \epsilon + \delta |u - u^*(\xi, \phi_\xi, \phi_S)|^2\ dt \\
&\leq -\int_0^{T_\Omega} c|(\dot{\xi}, \dot{S})|\ dt \\
&\leq -c \cdot \rho < 0
\end{aligned}$$

which holds for any policy for the maximizing player. This implies that

$$\sup_{u(\cdot)} \inf_{y(\cdot)} \phi(\xi(T_\Omega), S(T_\Omega)) < \phi(\bar{\xi}, \bar{S}) \quad .$$

Combining this with $\phi \geq U$ we obtain

$$\sup_{u(\cdot)} \inf_{y(\cdot)} U(\xi(T_\Omega), S(T_\Omega)) < U(\bar{\xi}, \bar{S})$$

which contradicts the dynamic programming principle. We conclude that the hypothesis $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) < 0$ cannot hold; in other words, U is a subsolution in the viscosity sense. ■

Remark 45: In the light of remark 42, it is instructive to consider $\min_i S_i$ as a candidate solution to to the Hamilton-Jacobi-Isaacs equation (4.19). First, $\min_i S_i$ is a viscosity *supersolution* as can readily be verified. Hence we can deduce a guaranteed cost strategy for y: at each instant y is chosen such that $\min_i S_i$ is non-increasing.

Second, $\min_i S_i$ is not *in general* a viscosity subsolution and therefore does not in general help us derive guaranteed cost strategies for u.

Third, $\min_i S_i$ is a *generalized* solution to (4.19) in the sense that the equation holds for almost all ξ, S (viz. whenever $\hat{\theta} = \arg\min_i S_i$ is well defined). This property is important in the following section where we discuss a *weak* certainty equivalence principle concerning the *heuristic* certainty equivalence controller. □

A theorem and a conjecture on the structure of U

Consider the canonical equations governing the *Hamiltonian* dynamics associated with U:

$$\dot{\xi}_i = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial H}{\partial \xi_i}, \quad \dot{S}_i = \frac{\partial H}{\partial \mu_i}, \quad \dot{\mu}_i = -\frac{\partial H}{\partial S_i} = 0 \quad . \tag{4.20}$$

It is well known (see e.g. [120]) that *if* the lower value function U is C^1, then the trajectories (ξ, S, λ, μ) corresponding to the saddle point strategies u^*,

y^*, solve the canonical equations. Hence the co-state μ is *constant* along the saddle point trajectories. Now $u^*(\xi,\lambda,\mu)$ and $y^*(\xi,\lambda,\mu)$ are linear in (ξ,λ) for fixed μ which implies that the trajectories also solve a *linear* system. Furthermore this linear system is the canonical equations associated with the *weighted* linear-quadratic game

$$Z(\xi(0),S(0);\alpha) = \sup_{u(\cdot)} \inf_{y(\cdot)} \sum_i \alpha_i S_i(\infty) \tag{4.21}$$

where $\alpha=\mu$.

This fits with the following observation: If a controller K is such that (Φ,K) dissipates $\dot{S}_i$ for $i=1,\ldots,p$, then (Φ,K) also dissipates $\sum_i \alpha_i \dot{S}_i$ for any non-negative weights α_i with $\sum_i \alpha_i = 1$ (proposition 2 on page 32).

This leads us to believe that the minimax controller at each instant chooses an *equivalent* linear-quadratic game, given by $\alpha=\mu$, and plays the minimax control of that game. In fact we have the following theorem:

Theorem 46: Assume that Z is finite everywhere and C^1, and that a differentiable function $\alpha^*(\xi,S)$ exists such that

$$Z(\xi,S;\alpha^*(\xi,S)) = \min_\alpha Z(\xi,S;\alpha) \quad .$$

Here minimization is over $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$. Then $U(\xi,S) = \min_\alpha Z(\xi,S;\alpha)$. Furthermore, the control law

$$u^U(\xi,S) = u^*(\xi,U_\xi(\xi,S),U_S(\xi,S))$$

guarantees that (P_θ,K) is has $\mathcal{L}_2$-gain less than or equal to γ for all θ, and that $x(t)\to 0$ as $t\to\infty$ for any $\mathcal{L}_2$ disturbances $w(\cdot)$, $v(\cdot)$. □

Proof: First, note that the one half of the statement $U=\min_\alpha Z$ is trivial:

$$U(\xi,S) = \sup_u \inf_y \min_\alpha \sum_i \alpha_i S_i(\infty) \leq \min_\alpha Z(\xi,S;\alpha) \quad .$$

To show that also the other inequality holds, we denote

$$\psi(\xi,S) = \min_\alpha Z(\xi,S;\alpha) = Z(\xi,S;\alpha^*(\xi,S))$$

and aim to show $\psi \leq U$ using proposition 41. First, take $\alpha=(1,0,\ldots,0)$; then $\psi(\xi,S) \leq Z(\xi,S;\alpha) = S_1$. Thus $\psi \leq \min_i S_i$.

Second, we must show that $H(\xi,S,\psi_\xi,\psi_S)\geq 0$. Here α^* being a minimizer implies that

$$\begin{aligned}
\psi_\xi(\xi,S) &= Z_\xi(\xi,S;\alpha^*(\xi,S)) + Z_\alpha(\xi,S;\alpha^*(\xi,S))\frac{\partial\alpha^*}{\partial\xi}(\xi,S)\\
&= Z_\xi(\xi,S;\alpha^*(\xi,S)) \quad ,\\
\psi_S(\xi,S) &= Z_S(\xi,S;\alpha^*(\xi,S)) + Z_\alpha(\xi,S;\alpha^*(\xi,S))\frac{\partial\alpha^*}{\partial S}(\xi,S)\\
&= Z_\xi(\xi,S;\alpha^*(\xi,S)) \quad .
\end{aligned}$$

Since Z solves the Hamilton-Jacobi-Isaacs equation $H(\xi,S,Z_\xi,Z_S)=0$ for each α, these expressions imply that also $H(\xi,S,\psi_\xi,\psi_S)=0$. Thus we can apply proposition 41 to show that $\psi\leq U$, hence $\psi=U$, and that the control law u^U guarantees $\min_i S_i(T)\geq U(\xi(0),S(0))$ for all T and inputs $y(\cdot)$. Finally combine with theorem 40 to see that this control applied to P_θ guarantees an $\mathcal{L}_2$ gain less than or equal to γ as well as internal stability. ■

The theorem provides the following solution to the simultaneous $\mathcal{H}_\infty$ control problem: First, construct the filter bank (4.13), (4.14) which generates the estimates ξ_i, S_i. Second, determine off-line the quadratic value functions $Z(\xi,S;\alpha)$ by finding the stabilizing solutions to a family of Riccati equation; one for each α. This yields the corresponding feedback controls

$$u^Z(\xi,S;\alpha) = u^*(\xi,Z_\xi(\xi,S;\alpha),Z_S(\xi,S;\alpha))$$

which are *linear* in ξ. Then, on-line, determine the minimizing argument α^* and apply the control $u^U(\xi,S)=u^Z(\xi,S;\alpha^*(\xi,S))$.

One could argue that this solution is only partial since differentiability of Z and α^* is *sufficient* but not *necessary* for the existence of a simultaneous $\mathcal{H}_\infty$ controller. Indeed, Z may take the value $+\infty$ for some values of ξ and α, and - more importantly - α^* may be discontinuous, when more than one minimizing argument of $\min_\alpha Z(\xi,S;\alpha)$ exist. At this point it is not clear how profound these difficulties are, and this topic deserves further attention. To this end, a good working hypothesis is the following:

Conjecture 47: The lower value function $U(\xi,S)$ is finite for all ξ, S if and only if $Z(\xi,S;\alpha) > -\infty$ for all ξ,S,α. In this case

$$U(\xi,S) = \min_\alpha Z(\xi,S;\alpha)$$

where minimization is over weights $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$. □

A result corresponding to the conjecture was stated recently in [90] for a finite horizon problem in discrete time; the proof has not been published. A complication related to the continuous-time setting is that discontinuities in the control law for u may lead to a closed loop system which is not well-posed.

In general, it adds some credibility to the conjecture is that many multi-objective optimization problems have been shown to be equivalent to weighted problems. A recent contribution concerning minimax control of a discrete system on a finite horizon is found in [89], which also contains further references. In order to prove the conjecture, or similar results, one probably needs to make use of viscosity solutions as well as minimax theorems [101], and investigate in further detail how the games associated with Z depend on α.

Summary of the discussion of the minimax controller

Let us briefly recapitulate our results concerning the minimax controller:

- The minimax controller is not based on certainty equivalence.
- The output feedback minimax control problem can be formulated as a full information problem of control for multi-dissipation (proposition 40). The control object in this equivalent problem is the filter bank (4.13), (4.14) which generates the minimax estimates of x, θ.
- The lower value function U of the corresponding game is not necessarily C^1, but solves a certain Hamilton-Jacobi-Isaacs equation in the viscosity sense (proposition 44). In addition, C^1 subsolutions of this equation generate guaranteed cost controllers, which solve the problem of simultaneous $\mathcal{H}_\infty$ control (proposition 41).
- These controllers can be implemented with the $p \times (n+1)$ states ξ, S.
- Theorem 46 reduces the task of solving the Hamilton-Jacobi-Isaacs equation by determining the structure of U: U can be derived from a study of the weighted optimization problems, providing that additional assumptions hold. Finally conjecture 47 suggests that these additional assumptions can be removed.

4.6 Heuristic certainty equivalence control

Even if the assumptions of theorem 46 are met, the resulting minimax controller is quite complex and requires substantial computation, both off-line and on-line. From a practical point of view it is therefore of great interest to investigate what can be obtained with simpler controller architectures.

In this section we consider the *heuristic certainty equivalence* controller

$$u(t) = u^{\hat\theta(t)}(t) = -B_{\hat\theta(t)} X_{\hat\theta(t)} \xi(\hat\theta(t), t) \tag{4.22}$$

where

$$\hat\theta(t) = \arg \min_i S_i(t) \tag{4.23}$$

as before. Notice that the control law has yet to be defined at points where the minimum $\min_i S_i$ is attained at more than one i. Controllers which are based on certainty equivalence, but without justifying certainty equivalence principles, are common in adaptive control [4].

The state of the controller is $(\xi(t), S(t))$. Recalling that the control objective is that the closed loop dissipates the supply rate s_θ as defined in equation (4.4), consider as a candidate control storage function

$$V(\theta, x(t), \xi(t), S(t)) = R(x(t), \theta, t) - \min_i S_i(t) \tag{4.24}$$

which may be computed as

$$V(\theta, x, \xi, S) =$$

$$\frac{1}{2}(x-\xi_\theta)'(Q_\theta - X_\theta)(x-\xi_\theta) + S_\theta + \frac{1}{2}x' X_\theta x - \min_i S_i \quad .$$

The candidate control storage function V is locally Lipschitz and hence differentiable almost everywhere, viz. wherever $\hat\theta = \arg \min_i S_i$ is well defined. Here the differential dissipation inequality holds, i.e.

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{2}\gamma^2|v|^2 + \frac{1}{2}\gamma^2|w|^2 - \frac{1}{2}|H_\theta x|^2 - \frac{1}{2}|u|^2 \\ &-\frac{1}{2}|\gamma w - \frac{1}{\gamma}G_\theta'(X_\theta x + (Q_\theta - X_\theta)(x-\xi_\theta))|^2 \\ &-\frac{1}{2}\gamma^2|v + C_\theta x - C_{\hat\theta}\xi_{\hat\theta}|^2 + \frac{1}{2}|u + B_{\hat\theta}' X_{\hat\theta}\xi_{\hat\theta}|^2 \quad . \end{aligned}$$

We see that whenever $\hat\theta(t)$ is well defined, the heuristic CE control $u(t) = u^{\hat\theta(t)}(t)$ is the *maximum dissipation* control law with which V indeed satisfies the differential dissipation inequality $\dot V \le s$. Thus V is a *generalized solution*[3] to the differential dissipation inequality, in the sense of [35, p. 20].

It is now straightforward to pose the following result:

Proposition 48: Let assumptions 30, 31, 34 and 35 hold, let the heuristic CE control law (4.22) be used and assume that $\hat\theta(t)$ is well defined by (4.23) almost everywhere on $[0,T]$. Then the $\mathcal{L}_2$ gain objective is met, i.e.

$$\frac{1}{2}\int_0^T |z|^2\, dt \le \frac{1}{2}\gamma^2\int_0^T |w|^2 + |v|^2\, dt + \frac{1}{2}x_0' Q_\theta x_0 + \Lambda_\theta \quad .$$

If furthermore $w \in \mathcal{L}_2([0,\infty))$, $v \in \mathcal{L}_2([0,\infty))$ and $\hat\theta(t)$ is well defined almost everywhere on $[0,\infty)$ then $x(t)\to 0$ as $t\to\infty$. $\triangle$

Proof: We have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \gamma^2|w|^2 + \gamma^2|v|^2 - |z|^2\, dt + \frac{1}{2}x_0'Q_\theta x_0 + \Lambda_\theta \\
&\ge R(x(T),\theta,T) \ge S(x(T),\theta,T) \ge \min_i S_i(T) \quad .
\end{aligned}$$

Since $\hat\theta(t)$ was assumed to be well defined for almost all $t\in[0,T]$ we have

$$\frac{d}{dt}\min_i S_i(t) = \frac{d}{dt}S_{\hat\theta(t)}(t) = \frac{1}{2}\gamma^2|y(t) - C_{\hat\theta(t)}\xi_{\hat\theta(t)}|^2 \ge 0$$

for almost all $t\in[0,T]$ due to the control law (4.22) and hence

$$\min_i S_i(T) \ge \min_i S_i(0) \ge 0$$

from which the result follows.

To show internal stability we follow the proof of proposition 39. ■

This result can be termed a *weak* certainty equivalence principle: Whereas the CE principle in [14] requires that the extended state estimate is *always* unique and concludes that the minimax controller is based on certainty

[3] V is also viscosity subsolution but in general not a supersolution which would imply dissipation [53]. Compare also with remark 45 above.

equivalence, the present result states that if the estimate is *almost always* unique, then the *heuristic* CE controller solves the original control problem, although it may not be minimax.

The condition is not completely satisfying since it imposes a restriction on the disturbances $w(\cdot)$ and $v(\cdot)$. One can draw a parallel to the assumption of persistent excitation in stochastic adaptive control: This condition is also not verifiable *a priori*, and a safety system must be added to the controller, so that proper action can be taken if the condition fails to hold. However, in contrast to the assumption of persistent excitation, it is difficult to see exactly which disturbances $w(\cdot)$, $v(\cdot)$ yield $\theta(t)$ being well defined almost everywhere, and hence it is difficult for the practicing engineer to judge if the restriction is reasonable. Further work on this issue is needed.

A smooth approximation of the controller

Since the control law is discontinuous at points where the minimum $\min_i S_i$ is attained for more than one i (indeed, the control law has yet not been defined at the points of discontinuity), some modification is needed to avoid chattering. Dwell-time switching or hysteresis switching are suggested in [79]. Here we consider as an alternative to approximate the control law with a smooth one. This will ease the load on the actuator hardware and prevent excitation of unmodeled fast dynamics. To this end, let us modify the candidate control storage function (4.24) to

$$\tilde V(\theta,x,\xi,S) = R(x,\theta,t) - f(S)$$

where the function f is the approximation of $\min_i S_i$ given by

$$f(S) = -\frac{1}{\eta}\log\left(\sum_{j=1}^p e^{-\eta S_j}\right) \quad . \tag{4.25}$$

Here $\eta > 0$ is a fixed parameter which determines the accuracy of the approximation. The function $f(S)$ enjoys the following properties which make it a suitable approximation of $\min_i S_i$: 1) f is C^∞, 2) f satisfies $\partial f/\partial S_i \ge 0$ and $\sum_i \partial f/\partial S_i = 1$, and finally 3) $f(S) < \min_i S_i < f(S) + \eta^{-1}\log p$.

The maximum dissipation controller corresponding to $\tilde V$ is

$$u^{\tilde V}(\xi,S) = \sum_{i=1}^p \frac{\partial f}{\partial S_i}(-B_i X_i \xi_i)\ ,$$

i.e., it is a weighted sum of the conditional minimax control suggested by each estimator. The derivation of this expression, as well as some further comments on this control law, can be found in [116].

Supervision of the controller

As mentioned above, the heuristic certainty equivalence controller should be supervised since we cannot prove that it guarantees satisfying operation. The dissipation analysis suggests that such a supervisory system should monitor the signals $S_i(\cdot)$. In particular, a decrease in $\min_i S_i$ or $f(S_i)$ indicates that the controller has not identified the plant and is uncertain about which control signal to actuate. On the other hand, a sudden increase in this signal should also attract attention as it indicates that the disturbances behave unexpected - a possible cause could be that a change in system parameters has occured.

We conclude the discussion of heuristic certainty equivalence control with a brief description of a simulation study:

Example 49: In [116] we discussed a case study regarding position control of an inverted pendulum, see figure 4.4. Here we briefly recapitulate the discussion; see [116] for further details.

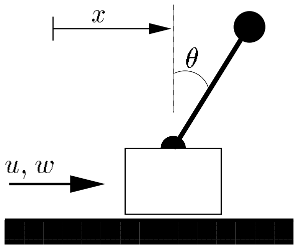

Figure 4.4: An inverted pendulum with force control and disturbance

The inverted pendulum is popular in benchmark problems because it is nonlinear, unstable, and minimum-phase (from the control force to the cart position), and yet relatively simple.

In our study, the plant is equipped with three sensors: One measuring the position of the cart, one measuring the angular position of the rod, and one measuring the angular velocity of the rod. The latter is subject to fault so we use two models to represent the control object; a nominal model and one corresponding to the sensor fault.

A simultaneous controller for the two *linearized* plant models is constructed, using the heuristic certainty equivalence architecture developed above.

The residuals S_i are pre-filtered with a first-order low-pass filter before the plant estimate $\hat{\theta}(t)$ is generated. This corresponds to exponential forgetting in adaptive control, [85, 4].

Simulation results with the nonlinear plant and the switching controller are shown in figures 4.5 and 4.6. Here a sensor fault occurs at time 7.4 seconds, which is at a critical stage after a step in the position reference. The fault is detected within approximately 0.2 seconds (figure 4.6). Some oscillations result from the fault but the system is rapidly stabilized (figure 4.5). After the fault has been detected system performance is worse since the one less sensor implies worse state estimates.

The residuals S_i seem to be quite well suited as indicators of model fit, and the heuristic certainty equivalence controller works nicely in this example. Although further work is needed with respect to forgetting schemes and modifications of the switching mechanism, the controller architecture seems to be reasonable and holds some promise. □

4.7 Conclusion

In this chapter we have applied nonlinear $\mathcal{H}_\infty$ theory to the problem of simultaneous $\mathcal{H}_\infty$ control of a finite number of linear plants. Our motivation for investigating this problem is that it appears to be the simplest problem of adaptive $\mathcal{H}_\infty$ control, if one excludes problems where parameter uncertainty is restricted to special system parameters.

We have shown that simultaneous $\mathcal{H}_\infty$ control involves a nonlinear $\mathcal{H}_\infty$ problem which possesses a number of simplifying features: The full information subproblem can be solved using linear theory. The cost-to-go, or the information state, is a quadratic function on state space which also can be found using linear theory. Although certainty equivalence does not apply, the simultaneous $\mathcal{H}_\infty$ control problem can be reduced to a state feedback problem on the worst-case filter, and hence be solved with finite-dimensional dynamic programming. However, these worst-case filters will

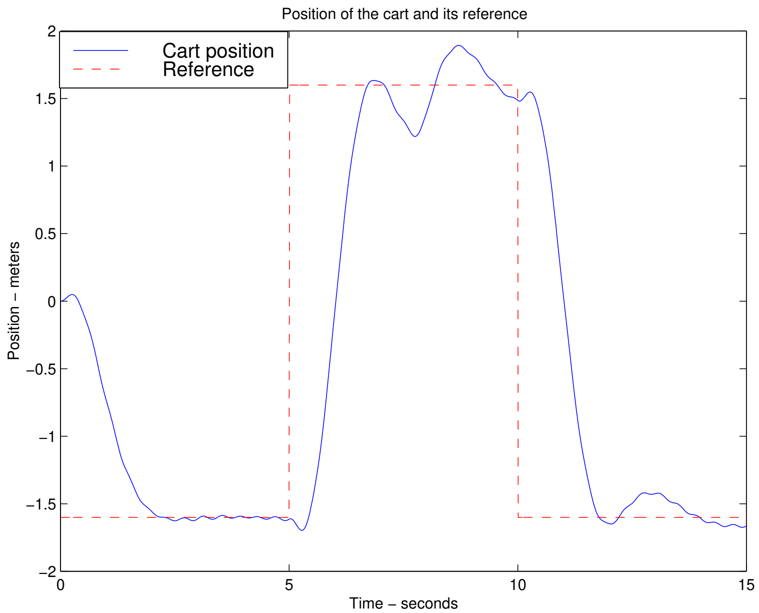

Figure 4.5: The position of the cart and its reference. Sensor fault in the angular velocity sensor at time 7.4 seconds.

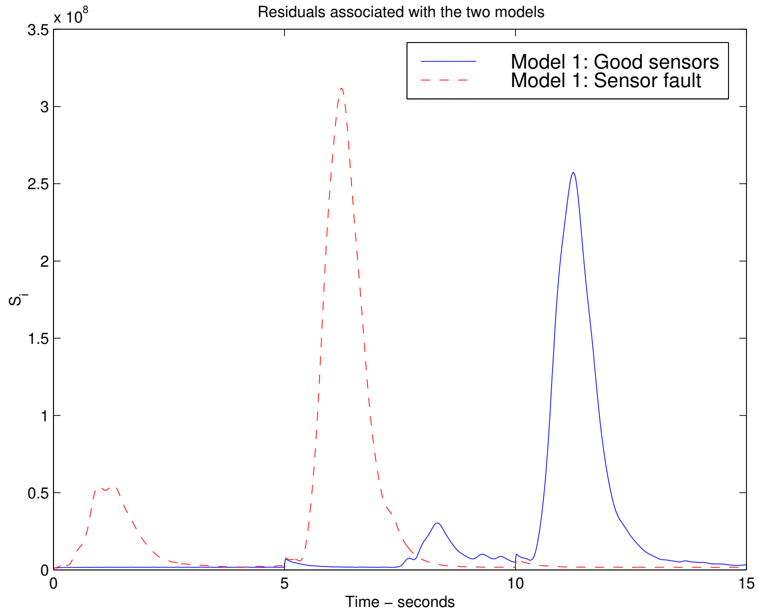

Figure 4.6: The residuals associated with the two models. The controller corresponding to the lower residual is connected to the plant. The fault at time 7.4 seconds is detected when the lines cross; approximately at time 7.6 seconds.

always be high-dimensional (example 49 leads to a filter with 14 states despite being somewhat academic) which makes direct numerical solution impossible and *a priori* insight into the structure of the solution necessary. We have made considerable progress in this direction, although a complete solution requires conjecture 47 to be verified a falsified. Furthermore we have investigated the heuristic certainty equivalence controller, and although the assumptions under which we can guarantee its performance are very restrictive, our simulations study suggests that its architecture is quite reasonable.

The work reported in this chapter may be continued in several directions: Further theoretical study of the problem may lead to conjecture 47 being resolved, or less restrictive conditions under which the heuristic certainty equivalence controller is sufficient. Also approximations of the heuristic certainty equivalence controller and investigation of various forgetting schemes is a subject which deserves more attention. Some hints towards other subjects are given in the succeeding notes.

Acknowledgements

The author wishes to thank Dr. A. Rapaport for stimulating discussions, and Prof. G. Vinnicombe for pointing out the work of S. Rangan and K. Poolla [90], which is described below.

4.8 Notes and references

Related recent literature

A study of simultaneous output feedback $\mathcal{H}_\infty$ control using digital controllers is presented by Savkin in [94]. The approach in this reference is reminiscent of the information state machinery, although it is not explicitly used since the problem is not formulated as a nonlinear $\mathcal{H}_\infty$ problem. The main result of the paper is that a feasible controller exists if and only if *a)* the filter algebraic Riccati equation (4.12) admits a suitable solution (although a small perturbation of the equation is necessary due to the use of digital controllers), *and b)* a full information minimax control problem, which except for the use of piecewise constant control signals is similar to

the problem discussed in section 4.5, admits a solution. As a consequence we may restrict attention to finite dimensional controllers. The problem of explicitly characterizing the solution of this state feedback minimax control problem is not addressed. It is interesting to notice that the use of digital controllers leads to a certain amount of technical simplification.

In [90] Rangan and Poolla consider a problem of simultaneous $\mathcal{H}_\infty$ control, which is similar to the one studied here, but is formulated in discrete time and on finite horizon. The approach is based on the information state machinery. Difficulties regarding regularity of the value function are avoided due to the finite horizon discrete time setting. Very interestingly, a result which resembles our conjecture 47 is stated. The proof of the result has not been published (the result is not included in [88]), and therefore it is not clear if it can be modified to assist in the verification of our conjecture 47. Other problems related to identification and control of a plant with multiple models are investigated in [88].

Jumping parameters

In applications one must usually expect that the parameter θ is not constant for all time, but will occasionally jump. This holds whether the problem is one of fault handling or an approximation to an adaptive control problem, where the continuous parameter space has been discretized.

As we mentioned in example 49 in section 4.6, one may add a exponential forgetting scheme to the heuristic certainty equivalence controller in order to make it handle parameter jumps. This forgetting scheme and others are popular in adaptive control [4, 85], and although it is most often difficult to carry through a rigorous analysis of the resulting system, experience indicates that they work quite well.

A rigorous approach to the problem with jumping parameters is to model the parameter variations with a Markov chain, which then leads to a stochastic dynamic game. The full information, finite time version of this game is treated in [8], where the solution is shown to be governed by a p coupled differential Riccati equations. The corresponding output feedback problem is open and involves several new difficulties, regarding the characterization and rôle of the information state.

Relaxing the simplifying assumptions

We have assumed the *simple case* [128] of the p linear $\mathcal{H}_\infty$ control problems, i.e. observability of (H_i, A_i), controllability of (A_i, G_i), decoupled process and measurement noise w and v, and a decoupled error signal $z = ((Hx)', u')'$. Relaxing these assumptions involves mainly algebraic manipulations [128], although certain details require attention.

Another assumption which can easily be removed is that the p plants have state space representations with the same dimension.

Part II

Stochastic models

Chapter 5

Dissipation in stochastic systems

We define the property of dissipativity for controlled Itô diffusions, and we investigate elementary properties, such as differential dissipation inequalities, convexity, and the connection to stability.

5.1 Introduction

Dissipative systems play a central rôle in the deterministic theory of robust stability, as evident from the works of numerous authors and also from the first part of this thesis. The deterministic theory also enables performance analysis, where performance is measured by the response to initial conditions, or by the worst-case response to an input in some set. The resulting framework has much appeal from a theoretical as well as from an engineering point of view, and is in accordance with the currently dominating paradigm for robust control.

A drawback of this framework is that it does not allow for stochastic representations of uncertainty, such as white noise disturbances, or for stochastic performance measures, such as risk of failure. On the other hand, the literature on stochastic systems does little to address the issues of robustness towards dynamic perturbations which motivated for instance the development of $\mathcal{H}_\infty$ control.

This suggests that it may be fruitful to extend the theory of dissipation to stochastic systems, and apply it to robustness analysis of stochastic systems. In this chapter we report results which indicate that the concept of dissipation is indeed meaningful in a stochastic context, and that much of the deterministic theory applies more or less directly.

Dissipation-like properties of stochastic systems do appear in the literature. For instance [31] uses stochastic Lyapunov functions to achieve bounds on the $\mathcal{L}_2$-gain of a wide sense linear system with deterministic inputs and stochastic outputs. Another example is the stochastic small gain theorem in [30] which connects input-output properties to Riccati equations, the solutions of which are subsequently used to obtain a stochastic stability result.

5.2 Preliminaries

We consider a controlled process x_t in a Euclidean state space $\mathbb{X} = \mathbb{R}^n$ given by an Itô stochastic differential equation evolving on the time interval $\mathbb{T} = [0, \infty)$

$$dx_t = f(x_t, w_t)\ dt + g(x_t, w_t)\ dB_t, \quad x_0 = x \in \mathbb{X} \tag{5.1}$$

where B_t is standard m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect a given filtration $\mathcal{F}_t$. The initial condition x is deterministic. The input w_t is an $\mathcal{F}_t$-adapted process taking values in Euclidean space $\mathbb{W} = \mathbb{R}^p$. See [83] for the necessary background material.

The system exchanges some quantity with its environment, specified by a supply rate $r : \mathbb{X} \times \mathbb{W} \to \mathbb{R}$. The accumulated flow from environment into the system during the time interval $[0, t]$ is R_t where

$$dR_t = r(x_t, w_t)\ dt, \quad R_0 = 0 \quad . \tag{5.2}$$

Notice that we here consider the supply to be a function of state and input, rather than a function of input and output. The motivation for this is simply to achieve a shorter notation, and the reader may substitute $r(x, w)$ with $s(z, w)$ if he so pleases, where $z = h(x, w)$ is the output.

We do not wish to dwell on technicalities regarding existence and uniqueness of solutions. Hence we simply restrict the input w_t to a set $\mathcal{W}$ of $\mathcal{F}_t$-adapted inputs for which there exists a unique t-continuous solution x_t, and assume that $\mathcal{W}$ is sufficiently large and closed under switching so that the principle of optimality holds.

Associated with the equation (5.1) we define for each $w \in \mathbb{W}$ the differential operator $L^w : C^2(\mathbb{X}, \mathbb{R}) \to C^0(\mathbb{X}, \mathbb{R})$ given by $L^w V(x) = V_x f + \frac{1}{2}\mathrm{tr} g' V_{xx} g$ where the right hand side is evaluated at (x, w).

If J is a functional on sample paths of the processes x_t, w_t, then $E^x J$ is expectation w.r.t. the probability measure generated by x_t, w_t with initial condition $x_0 = x$. In this notation the dependence of $E^x J$ on the input w_t is suppressed.

5.3 Definition of dissipativeness and elementary properties

Recall that the fundamental element in the deterministic theory of dissipation [124] is the storage function $V : \mathbb{X} \to \mathbb{R}$ which satisfies the dissipation inequality

$$V(x_t) \leq V(x_0) + \int_0^t r(x_s, w_s)\ ds$$

along every trajectory of the system. This inequality can be generalized to a stochastic setting in several ways, but it appears that the most useful framework is achieved by requiring the inequality to hold in expectation:

Definition 50: We say that the system (5.1) is *dissipative* w.r.t. the supply rate r, if there exists a non-negative *storage function* $V : \mathbb{X} \to \mathbb{R}$ such that the *integral dissipation inequality*

$$E^x\{V(x_\tau) - \int_0^\tau r(x_s, w_s)\ ds\} \leq V(x) \tag{5.3}$$

holds for all bounded stopping times τ and all solutions x_t, w_t of the system with $x_0 = x \in \mathbb{X}$. □

We emphasize that the dissipation inequality is only required to hold for bounded stopping times τ; see p. 114 below for a comment.

Using the results in e.g. [83], it is easy to see that storage functions are related to a differential version of the dissipation inequality:

Proposition 51: A nonnegative C^2 function $V : \mathbb{X} \to \mathbb{R}$ is a storage function if and only if it satisfies the differential dissipation inequality

$$\sup_{w\in\mathbb{W}} L^w V(x) - r(x,w) \leq 0 \tag{5.4}$$

on $\mathbb{X}$. △

Proof: Sufficiency: Let V be C^2 and satisfy the inequality (5.4). Let x_t, w_t be a solution with $x_0 = x \in \mathbb{X}$ and let τ be a bounded stopping time, i.e. $\tau < T$. Let $(x_{t\wedge\tau}, R_{t\wedge\tau})$ be the process (x_r, R_t) stopped at τ, i.e.

$$dx_{t\wedge\tau} = dx_t \cdot \chi_{t\leq\tau}, \quad dR_{t\wedge\tau} = dR_t \cdot \chi_{t\leq\tau}$$

(Here $\chi_{t\leq\tau}$ is the indicator function, i.e. $\chi_{t\leq\tau} = 1$ if and only if $t \leq \tau$ and 0 otherwise). Now consider $V(x_{t\wedge\tau}) - R_{t\wedge\tau}$. By Itô's lemma this process is again an Itô process and the differential dissipation inequality implies $E^x(V(x_{t\wedge\tau}) - R_{t\wedge\tau}) \leq V(x)$. Now notice that $x_{T\wedge\tau} = x_\tau$ and $R_{T\wedge\tau} = R_\tau$; we have thus shown that the inequality (5.3) holds.

Necessity: Let V be a C^2 storage function and consider a solution (x_t, w_t) for which the input is constant and deterministic, $w_t \equiv w \in \mathbb{W}$, and the initial condition $x_0 = x \in \mathbb{X}$ is deterministic. Then V is in the domain of the characteristic operator $\mathcal{A}^w$ (see [83, p. 116]) and the dissipation inequality (5.3) implies that $L^w V(x) = \mathcal{A}^w V(x) \leq r(x,w)$. Since x and w were arbitrary the conclusion follows. ■

We define the available storage of the system (5.1) w.r.t. the supply rate r in a manner analogous to [124], namely by

$$V_a(x) = \sup_{w_t,\tau} E^x \int_0^\tau -r\ ds \tag{5.5}$$

where the supremum is over all bounded stopping times τ and all solutions x_t, w_t with $x_0 = x$. With this definition we immediately have a result analogous to theorem 1 in [124, p. 328]:

Proposition 52: The available storage is finite for all $x \in \mathbb{X}$ if and only if the system is dissipative. Furthermore, in this case the available storage is in itself a storage function and any other storage function V satisfies

$$V(x) \geq V_a(x), \quad \forall x \in \mathbb{X} \quad .$$

Finally $\inf\{V_a(x) : x \in \mathbb{X}\} = 0$. △

Proof: First we show that if the available storage is finite, then it is a storage function. It is immediate that $V_a \geq 0$ (if necessary, this is obtained by letting $\tau \to 0$). The dissipation inequality then reads

$$E^x\{V_a(x_\tau) - \int_0^\tau r\ ds\} \leq V_a(x)$$

which follows from the principle of optimality. Hence the system is dissipative.

Second we show that if the system is dissipative with storage function V, then we have $V_a \leq V$; in particular the available storage is finite. To see this we rewrite the dissipation inequality as

$$E^x \int_0^\tau -r\ ds \leq V(x) - E^x V(x_\tau) \leq V(x)$$

where the second inequality follows from V being non-negative. Since this inequality holds for all bounded stopping times τ and all solutions x_t, w_t which satisfy $x_0 = x$ we have $V_a(x) \leq V(x) < \infty$. The conclusion follows.

Finally we show the last claim: Let $V(x)$ be a storage function, then it is easy to see that so is $V(x) - \inf_\xi V(\xi)$, hence $V_a(x) \leq V(x) - \inf_\xi V(\xi)$. It follows that $\inf_x V_a(x) \leq 0$. ■

The available storage is related to a differential dissipation *equality*; see the note on page 116 below.

In chapter 2 on deterministic systems, we stated that storage functions and supply rates satisfy a joint convexity property (proposition 2 on page 32). This generalized a statement of Willems [124, theorem 3, p. 331] and was the key to the chapter 3, which reduced robustness analysis to convex optimization. This approach to robustness analysis is also fruitful in a stochastic context, which is the subject of the succeeding chapter. At this point we state a result similar to the deterministic proposition 2:

Proposition 53: Given a diffusion (5.1), a linear space $\mathcal{V}$ of candidate storage functions $V : \mathbb{X} \to \mathbb{R}$ and a linear space $\mathcal{R}$ of supply rates. Then the subset

$$\{(V,r) \subset \mathcal{V} \times \mathcal{R} \mid V \geq 0 \text{ and } (V,r) \text{ satisfy } (5.3)\}$$

is a convex cone. △

Proof: Let $r_i \in \mathcal{R}$ for $i = 1, 2$ be chosen such that the system is dissipative w.r.t. r_i and let two corresponding storage functions be $V(x; r_i)$. Let x_t, w_t be a solution with $x_0 = x \in \mathbb{X}$ and let τ be a bounded stopping time; we then know that

$$E^x\{V(x_\tau; r_i) - \int_0^\tau r_i \, dt\} \le V(\bar{x}; r_i) \quad .$$

By multiplying these two inequalities with positive constants α_i and adding the results we see that $\alpha_1 V(x; r_1) + \alpha_2 V(x; r_2)$ is a storage function for the system w.r.t. the supply rate $\alpha_1 r_1 + \alpha_2 r_2$. ∎

In particular, the set of dissipated supply rates in $\mathcal{R}$ is a convex cone, as is the case for deterministic systems (see chapter 2 or [45]). A related fact is the following:

Proposition 54: Let $V_a(x; r) \in [0, \infty]$ be the available storage of the system (5.1) with respect to the rate $r \in \mathcal{R}$, then for each x the function $V_a(x; r)$ is convex in r. △

Proof: The available storage is for each x defined as the supremum of a family of functionals which are convex in r; the same holds therefore for $V_a(x, \cdot)$. ∎

5.4 Linear systems and quadratic supply rates

Consider a homogeneous wide sense linear system

$$dx_t = [Ax_t + Bw_t] \, dt + \sum_{i=i}^{m} [F_i x_t + G_i w_t] \, dB_t^i \tag{5.6}$$

with a quadratic supply rate $r(x, w) = (x' \; w')Q(x' \; w')'$. We assume that r is concave-convex in (x, w) which implies that r is regular in the sense $r(x, 0) \le 0$. This system is linear in the sense that the set of solutions (x_t, w_t) is a linear space; in other words, the map from input process w_t and initial condition x to the state process x_t is linear. It can be shown that if such a system is dissipative then the available storage is a quadratic function of the initial state x, i.e. may be written as

$$V_a(x) = x' P_a x$$

where $P_a = P_a' \ge 0$. Furthermore, the quadratic storage functions $V(x) = x'Px$ with $P = P'$ are exactly those that satisfy $P \ge 0$ and the differential dissipation inequality (5.4) which can be rewritten as the linear matrix inequality

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} + \sum_{i=1}^{m} [F_i, G_i]' P [F_i, G_i] \le Q \quad . \tag{5.7}$$

It is thus possible to use LMI solvers as [38, 32] to answer the analysis questions: Is the system dissipative? If yes, what is the available storage?

Remark 55: It is well known (see e.g. [76] and the references therein) that multiplicative noise terms F_i, G_i can be advantageous for a linear system from the point of view of stability in probability. But such a noise term will always contribute positively to the left hand side of the inequality (5.7) which shows that multiplicative noise terms are always disadvantageous in analysis of dissipation w.r.t. a quadratic supply rate. □

Supply rates of special interest are those corresponding to passivity and small gain. Stochastic $\mathcal{L}_2$ gains have recently received some attention and stochastic bounded real lemmas as well as other results can be found in [30, 31, 48]. Stochastic passivity has, to our knowledge, not been considered in the literature, probably because stochastic passivity of an isolated system is of no particular interest. However, if a nominal stochastic system is connected to an unknown passive perturbation, then it is of great relevance if the nominal system is stochastically passive. We will later return to such robustness issues; at this point we state a stochastic positive real lemma:

Proposition 56: For the system (5.6), let the supply rate be $r(x, w) = 2\langle w, z\rangle$ with $z = Cx + Dw$. Then the following are equivalent:

1. The system is stochastically strictly input passive, i.e. stochastically dissipative w.r.t. $r - \epsilon|w|^2$ for some $\epsilon > 0$, and the autonomous system obtained with $w = 0$ is exponentially mean square stable.

2. There exists a $P = P' > 0$ such that

$$\begin{bmatrix} PA{+}A'P & PB \\ B'P & 0 \end{bmatrix} + \sum_{i=1}^{m} [F_i, G_i]' P [F_i, G_i] < \begin{bmatrix} 0 & C' \\ C & D'{+}D \end{bmatrix} \, . \tag{5.8}$$

△

Before we prove the proposition it is convenient to state an elementary matrix lemma:

Lemma 57: Let $Q = Q' > 0$, $R = R' > 0$, S and $T = T'$ be of compatible dimensions, then for $\alpha > 0$ sufficiently small the matrix inequality

$$\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} > \alpha \begin{bmatrix} -R & S \\ S' & T \end{bmatrix}$$

holds. □

Proof: [of the lemma] By Schur complement, the inequality holds if and only if $\alpha R > 0$ and $Q - \alpha T - \alpha S' R^{-1} S > 0$. These conditions are satisfied for $\alpha > 0$ small enough since $Q > 0$ and $R > 0$. ■

Proof: [of the proposition] To see that the linear matrix inequality condition is sufficient one needs only verify that $V(x) = x'Px$ is a storage function function w.r.t. $r - \epsilon|w|^2$ for sufficiently small ϵ, and that, with $w = 0$, V serves as a stochastic Lyapunov function to show exponential mean square stability using a standard sufficient condition [43, p. 200].

To show necessity we use that exponential mean square stability implies [43, p. 201] the existence of a $Z = Z' > 0$ such that

$$ZA + A'Z + \sum_{i=1}^{m} F^{i'} Z F^i < -\delta\ I$$

for some $\delta > 0$. Now let $\epsilon > 0$ and let $V(x) = x'Xx$ be a storage function for the system with respect to the supply rate $r - \epsilon|w|^2$, i.e. $X = X' \geq 0$ and

$$\begin{bmatrix} XA + A'X & XB \\ B'X & 0 \end{bmatrix} + \sum_{i=1}^{m} [F_i, G_i]' X [F_i, G_i] \leq \begin{bmatrix} 0 & C' \\ C & D' + D - \epsilon\ I \end{bmatrix} .$$

We claim that $P = X + \alpha Z$ solves the linear matrix inequality (5.8) for $\alpha > 0$ sufficiently small. To see this, insert $P = X + \alpha Z$ in (5.8) and reduce terms using the LMIs which X and Z satisfy, thus obtaining

$$\begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} > \alpha \begin{bmatrix} -\delta I & ZB + \sum F_i' Z G_i \\ B'Z + \sum G_i' Z F_i & \sum G_i' Z G_i \end{bmatrix} .$$

This inequality holds for $\alpha > 0$ small enough according to lemma 57 which completes the proof. ■

Apart from strict input passivity, one could imagine several other definitions of strict positive realness of a stochastic system, just as in the deterministic case [123].

5.5 Stability and interconnections of dissipative systems

As in proposition 56 and in deterministic theory [124, 45] a storage function often serves as a Lyapunov function to show that the isolated system is stable. Indeed, this is one of the properties which make dissipative systems interesting from a control point of view.

In order to investigate stability of the autonomous system

$$dx_t = f(x_t, 0)\ dt + g(x_t, 0)\ dB_t \tag{5.9}$$

we use the terminology of Has'minskiĭ [43]:

Definition 58: A constant solution $x_t \equiv \bar{x}$ of the autonomous equation (5.9) is stable in probability if for any $\epsilon > 0$

$$\lim_{x \to \bar{x}} P^x \{\sup_{t \geq 0} |x_t - \bar{x}| > \epsilon\} = 0$$

where the diffusion x_t solves (5.9) with $x_0 = x$. □

Using the existing Lyapunov-type criterion for stochastic stability [43] we immediately get the following:

Theorem 59: Let the supply rate r be regular in the sense that $r(x, 0) \leq 0$ for all x. Let the system (5.1) be dissipative with respect to r and let V be a continuous storage function which attains an isolated local minimum at $\bar{x} \in \mathbb{X}$. Then the process $x_t \equiv \bar{x}$ is a solution of the autonomous equation (5.9) and is stable in probability. □

Proof: The proof runs along the same lines as theorem 3.1 in [43, p. 164], the only deviation being that V is not required to be C^2 around $\bar{x}$. Let $a = V(\bar{x})$, let Ω be a neighbourhood of $\bar{x}$ such that $a < V(x)$ for $x \in \bar{\Omega} \backslash \{\bar{x}\}$. Let τ be the stopping time $\tau = \inf\{t :\ x_t \notin \Omega\}$. It follows from the dissipation inequality that $V(x_{t\wedge\tau}) - a$ is a supermartingale for any initial condition $x \in \Omega$. In particular if $x = \bar{x}$ then $x_t = \bar{x}$ w.p. 1 for all t which proves the first claim. Furthermore for $x \neq \bar{x}$ the supermartingale inequality of Doob (see e.g. [83, p. 28]) yields

$$P^x \{\sup_{t \geq 0} V(x_{t\wedge\tau}) - a \geq \epsilon\} \leq \frac{V(x) - a}{\epsilon}$$

which holds for all ϵ. Now pick arbitrarily small $\epsilon, \epsilon' > 0$ such the ϵ-ball around $\bar{x}$ is contained in Ω. We must show that there exists $\delta > 0$ such that $|x - \bar{x}| < \delta$ implies $P^x\{\sup_{t\geq 0} |x_t - \bar{x}| > \epsilon\} \leq \epsilon'$. To this end choose $V_2 > a$ such that $\xi \in \Omega$ and $V(\xi) < V_2$ together imply $|\xi - \bar{x}| < \epsilon$. Then choose $V_1 > a$ such that $(V_1 - a)/(V_2 - a) \leq \epsilon'$. Finally choose $\delta \in (0, \epsilon)$ such that $|\xi - \bar{x}| < \delta$ implies that $V(\xi) < V_1$. We then have the implications

$$
\begin{aligned}
|x - \bar{x}| < \delta \quad &\Rightarrow \quad V(x) - a < V_1 - a \\
&\Rightarrow \quad P^x\{\sup_{t\geq 0} V(x_{t\wedge\tau}) - a > V_2 - a\} \leq \frac{V_1 - a}{V_2 - a} \\
&\Rightarrow \quad P^x\{\sup_{t\geq 0} |x_t - \bar{x}| > \epsilon\} \leq \epsilon'
\end{aligned}
$$

as desired. ■

Remark 60: We say that the system (5.1) is *locally* dissipative around $\bar{x}$ w.r.t. the supply rate r if there exists a non-negative V and a bounded neighbourhood Ω of $\bar{x}$ such that the dissipation inequality holds provided $x_t \in \Omega$ for $0 \leq t < \tau$. In this case we say that V is a local storage function. A necessary and sufficient condition for a non-negative C^2 function V to be a local storage function is that it satisfies the differential dissipation inequality (5.4) on Ω. It is easy to see that the above theorem holds if the storage function V is replaced with a local storage function. □

One may show other stability properties such as stochastic sample path boundedness or exponential p-stability by imposing additional constraints on the storage function and the supply rate and using the corresponding Lyapunov-type theorems in [43].

As in the deterministic case, the stability implications of dissipativity is important in robustness analysis since systems consisting of dissipative components are themselves dissipative. Consider the simple case of two systems

$$\Sigma_i \;:\; dx^i = f^i(x^i, w^i)\; dt + g^i(x^i, w^i)\; dB^i$$

connected in feedback through the equations

$$w^1 = h^2(x^2, w^2) + v^1 \quad \text{and} \quad w^2 = h^1(x^1, w^1) + v^2 \quad .$$

Here h^i are output functions and v^i are exogenous inputs. Assume that each system is dissipative w.r.t. the rate $r^i(x^i, w^i)$. In addition, assume that the interconnecting equations have unique solutions $w^i = \bar{w}^i$ for all x^i and v^i (for instance, if one of the two h^i is independent of w^i) and that the resulting system satisfies the well-posedness assumptions of section 5.2 (in particular, (B^1, B^2) is standard Brownian motion w.r.t. the filtration $\mathcal{F}_t$). It is now easy to verify that the interconnection is dissipative w.r.t. the supply rate $r(x^1, x^2, v^1, v^2) = r^1(x^1, \bar{w}^1) + r^2(x^2, \bar{w}^2)$. Combining with the stability result of theorem 59 we get:

Proposition 61: Assume that the each of the storage functions $V^i(x^i)$ is continuous and attains an isolated local minimum at $x^i = 0$. Assume in addition that the supply rates satisfy $r(x^1, x^2, 0, 0) \leq 0$ for all x^1, x^2. Then $x^i_t \equiv 0$ is a solution of the interconnected system with $v^i_t \equiv 0$ and this solution is stable in probability. △

The main application of this result is to give a sufficient condition for *robust stability* of a stochastic system subject to a deterministic dissipative perturbation, for instance combining with the positive real lemma of proposition 56:

Corollary 62: Let a system Σ be given by the dynamics (5.6) and the output equation $z = Cx + Dw$, and let Σ be connected in feedback with a perturbation $\Delta : z \to w$ which is dissipative w.r.t. $-2\langle w, z\rangle$. Let the interconnection be well posed and let Δ possess a continuous storage function of which some point ξ is an isolated minimum point. Assume that there exists a $P = P' > 0$ such that the linear matrix inequality (5.8) holds. Then the constant process $(0, \xi)$ is a solution of the interconnection and this solution is stable in probability. □

The corollary demonstrates that, as in the deterministic theory, robustness questions can be resolved by computing storage functions; in the case of linear systems this reduces to linear matrix inequalities.

5.6 Chapter conclusion

It can be argued that the concept of dissipation in dynamical systems is the unifying factor behind a broad range of results in deterministic control theory, in particular within robust control. We believe that the appeal of the framework is not lost in the transfer to a stochastic context.

Although this chapter demonstrates that several key features of the deterministic theory generalizes to the stochastic setting, the stochastic theory is far from complete. Some comments on remaining problems are discussed in the following.

5.7 Notes and references

Unbounded stopping times

In our definition of a dissipative stochastic system, the integral dissipation inequality (5.3) was required to hold for bounded stopping times only. This leads to the question: If V is a storage function for the diffusion (5.1) and τ is an unbounded stopping time, does the dissipation inequality (5.3) hold?

The short answer to this question is: Not necessarily. Let $\epsilon > 0$, then a trivial counterexample is the stopping time

$$\tau = \inf\{t > 0 \ : \ V(x_t) - \int_0^t r \ dt > V(x) + \epsilon\}$$

for which

$$E^x V(x_\tau) - E^x \int_0^\tau r \ dt = V(x) + \epsilon$$

provided that V is continuous, implying that the dissipation inequality does not hold. It is possible to construct examples where this stopping time is finite almost surely - the interested reader is encouraged to consider the diffusion $dx_t = -x_t \ dt + w_t \ dB_t$ with the supply rate $r = -2x^2 + w^2$ and take the input w_t to be a non-zero constant.

A first step towards a more complete answer is that a sufficient condition for the dissipation inequality (5.3) to hold is that V is a storage function and that the family

$$\{V(x_{t\wedge\tau}) - \int_0^{t\wedge\tau} r \ dt\}_{t>0}$$

of random variables is uniformly integrable. This follows from a convergence result for uniformly integrable random variables, [83, p. 41] - we refer to the same reference for the definition of uniform integrability. We expect that more explicit results can be obtained for special classes of unbounded stopping times, such as the first exit time of x_t from a given domain.

Non-smooth storage functions and viscosity solutions

In deterministic theory of dissipation, it has been shown by James [53] that locally bounded storage functions can without loss of generality be taken to be lower semi-continuous (l.s.c.), and that l.s.c. storage functions are exactly the viscosity solutions to the differential dissipation inequality (5.4).

The question is if the analogous statements hold in the stochastic setting. It is easy to show that l.s.c. storage functions are indeed viscosity solutions to (5.4). We conjecture that also l.s.c. viscosity solutions to (5.4) are storage functions. Existing stochastic verification theorems in the framework of viscosity solutions [62, 131] are based on uniqueness results for viscosity solutions and are therefore not applicable to dissipation inequalities (or even the corresponding equalities) which have many solutions. The deterministic technique in [53] could probably be modified and applied; the additional complication that the dissipation inequality must hold for any *random* bounded stopping time τ could be addressed with the results on optimal stopping in [84].

Further questions are if locally bounded storage functions can taken to be l.s.c. and under what conditions they can be taken to be continuous or even C^2. These issues are left for future research.

The required supply

Recall that we in chapter 2 defined the required supply of a dissipative deterministic system as

$$V_r(x) = \inf_{x(\cdot),w(\cdot),T} \int_0^T r(t) \ dt$$

where the infimum is subject to the system dynamics and the conditions $V_a(x(0)) = 0$ and $x(T) = x$. We see that this definition does not extend directly to stochastic systems, because the presence of noise may make it impossible to reach a specified terminal state in finite time.

An alternative starting point for a definition is

$$V_r(x) = \sup_V V(x)$$

where the supremum is over all l.s.c. storage functions V for which $V(\xi) = 0$ whenever $V_a(\xi) = 0$. Assume that the required supply defined in this fashion is finite; then it is l.s.c. and satisfies

$$E^x\{V_r(x_\tau) - \int_0^\tau r\ ds\} \leq 0 \quad \text{if} \quad V_a(x) = 0 \quad .$$

Does the available storage satisfy a PDE?

It is well known that there is an intimate connection between the Hamilton-Jacobi-Bellman *equations* and the available storage, the required supply and other value functions, [83, 7]. Nevertheless, the exact nature of this connection is often misquoted, in that situations where the value function does not satisfy the Hamilton-Jacobi-Bellman equation are regarded as pathological. Consider as an example passivity analysis of a scalar wide-sense linear diffusion:

$$dx_t = (-x_t + w_t)\ dt + \sigma x_t\ dB_t\ , \quad r(x,w) = xw\ ,$$

where $\sigma \geq 0$ is a parameter. Since the system is linear and the supply rate is quadratic we know that the available storage is a quadratic function of the state. It is easy to verify that a quadratic storage function $V(x) = \alpha x^2$ must satisfy

$$\forall x, w:\ 2\alpha x(-x+w) + \alpha\sigma^2 x^2 \leq xw$$

which implies

$$\alpha = \frac{1}{2}, \quad \sigma^2 \leq 2 \quad .$$

We see that the system is dissipative if and only if $\sigma^2 \leq 2$. In this case the available storage satisfies the Hamilton-Jacobi-Bellman *inequality*

$$\sup_w \{L^w V_a(x) - r(x,w)\} = (-1 + \frac{1}{2}\sigma^2)x^2 \leq 0 \quad .$$

Only when $\sigma^2 = 2$ does the available storage satisfy the Hamilton-Jacobi-Bellman *equation*. The available storage solves a *strict* HJB-inequality when $\sigma^2 < 2$, for instance in the deterministic situation $\sigma = 0$.

The reason why the value function does not satisfy the HJB-equation is that no optimal solution exists. For the optimal control problem associated with the available storage, almost-optimal Markov controls are $w = -fx$ where $f \to +\infty$. Theorems which state that the value function satisfies a PDE (as for instance theorem 11.1 in [83]) need the existence of an optimal pair (x_t^*, w_t^*) (either explicitly assumed or implied by other assumptions) since their proofs involve differentiating the value function along x_t^*.

Another situation where the available storage does not satisfy the Hamilton-Jacobi-Bellman equation is when the supply rates are not regular, i.e. in some region of state space the input is forced to deliver a positive supply to the system. A trivial example is the system above with the supply rate 1. In general, non-regular supply rates lead to many contra-intuitive phenomena and should be treated with care or avoided.

Computation of storage functions with convex optimization

Consider the input-affine controlled diffusion on $\mathbb{X} = \mathbb{R}^n$

$$dx_t = (f(x) + \phi(x_t)\ w_t)\ dt + (g(x_t) + \gamma(x_t)\ w_t)\ dB_t$$

with the input-quadratic supply rate

$$r(x,w) = h(x) + 2k(x)\ w + w'\ l(x)\ w \quad .$$

We assume that both B_t and w_t are scalar processes. The case of vector processes is conceptually the same but the notation is more involved. The backwards operator is

$$L^w V(x) = V_x f + V_x \phi\ w + \frac{1}{2}(g + \gamma\ w)' V_{xx}(g + \gamma\ w)$$

for $V \in C^2(\mathbb{X}, \mathbb{R})$; we have omitted the argument x on the right hand side. The differential dissipation inequality (5.4) can then be written more explicitly as

$$P(V,x) := \begin{bmatrix} V_x f + \frac{1}{2}g' V_{xx} g - h & \frac{1}{2}V_x\phi + \frac{1}{2}g' V_{xx}\gamma - k \\ \frac{1}{2}(V_x\phi)' + \frac{1}{2}\gamma' V_{xx} g - k' & \gamma' V_{xx}\gamma - l \end{bmatrix} \leq 0 \quad . \tag{5.10}$$

Here $P : C^2(\mathbb{X}, \mathbb{R}) \times \mathbb{X} \to \mathbb{R}^{2\times 2}$. A non-negative C^2 function is a storage function if and only if this matrix inequality holds at each point x in state space (proposition 51 on page 106).

We now suggest the following numerical strategy for computing storage functions: Choose a set of basis functions $V^i \in C^2(\mathbb{X},\mathbb{R})$ and search for a storage function of the form

$$V(x) = \sum_{i=1}^{N} \alpha_i V^i(x) \quad .$$

The basis functions V^i could for instance be polynomials, trigonometric functions, or wavelets. In order to verify if V is a storage function, we test for dissipation and non-negativity at a set of points x^j, $j = 1, \ldots, M$. This leads to the LMI problem

$$\text{Find } \alpha_1, \ldots, \alpha_N \text{ such that}$$

$$\sum_{i=1}^{N} \alpha_i P(V^i, x_j) \leq 0 \ , \quad \sum_{i=1}^{N} \alpha_i V^i(x_j) \geq 0 \ \text{ for } j = 1, \ldots, M \tag{5.11}$$

for which software such as [38, 32] can find a solution or determine that no solution exists.

The LMI problem has N scalar variables, M scalar constraints and M 2-by-2 matrix constraints. If w is a m-vector rather than a scalar, then the matrix constraints will be $(m+1)$-by-$(m+1)$. Notice that the dimension of the state space does not affect the size of the matrices; however high-dimensional state spaces need a large number of basis functions V^i and a large number of evaluation points x_j in accordance with the curse of dimensionality.

If the differential dissipation inequality (5.10) is merely satisfied at points x_j, it is quite likely that it fails near x_j. Therefore, one may wish to consider strict inequalities in (5.11) and attempt to solve

$$\text{Find } \alpha_1, \ldots, \alpha_N, \beta_1, \beta_2 \text{ such that}$$

$$\sum_{i=1}^{N} \alpha_i P(V^i, x_j) \leq -\beta_1 \ \kappa(x) \ , \quad \text{for } j = 1, \ldots, M \quad ,$$

$$\sum_{i=1}^{N} \alpha_i V^i(x_j) \geq \beta_2 \ \lambda(x) \ , \quad \text{for } j = 1, \ldots, M \quad ,$$

$$\beta_1 > 0 \ , \quad \beta_2 > 0$$

where κ and λ are given functions.

Computing storage functions with LMI software is a relatively flexible principle which may be modified in several ways, depending on the specific application. For instance, one may search simultaneously for a supply rate in some convex polytope, add constraints on the storage function, its gradient or curvature, or one may include a linear functional of storage function and supply rate to be minimized.

If one goes beyond input-affine systems with input-quadratic supply rates, then storage functions may still be found with convex optimization but with much greater difficulty since the differential dissipation inequality does not reduce to LMIs in state space. Further complications arise when the supremum over w in the differential dissipation inequality (5.4) cannot be evaluated explicitly.

While the above discussion may be sufficient for illustrative academic case studies, it would be necessary for real-world applications to consider the numerics in greater detail. A specific question which deserves attention concerns the dual to (5.10). As emphasized in [19], when convex optimization is used as computational tool in control theory, the dual problem often have interesting control theoretic interpretations. See [57] for an example where dualism is utilized in discretized infinite-dimensional convex optimization problems.

Another strategy for numerical computation of storage functions is to solve a partial differential equation corresponding to the differential dissipation inequality (5.4) using a finite difference scheme [65]. The two approaches, convex optimization and numerical solution of PDEs, may also be combined.

Simplifying computations with modularity

For a realistic problem involving more than a couple of states and without specific simplifying structure, the approaches outlined above become unrealistic as the numerical burden becomes overwhelming. In this case it may be feasible to decompose the system into a number of sub-systems. These sub-systems need not correspond to physical units but could for instance be dynamic modes which are known to interact weakly. Then one may perform dissipation analysis on each of the subsystems and after this conclude on the dissipation of the overall system using the results on interconnections of dissipative components. In effect this corresponds to imposing a specific structure on the storage function of the overall system. Needless to say, the effectivity of this approach relies heavily on the physical and mathematical insight into the system.

For deterministic systems with (single) supply rates corresponding to passivity, this approach to analysis goes back to Popov's work on hyperstability [87]. An interesting topic of future research would be systematic modularization. Backstepping and other recursive design techniques [63, 24] can be seen as extreme examples of systematic modularization.

Stratonovich equations

In this dissertation we work exclusively with the Itô interpretation of stochastic differential equations. In some applications it is more natural to model uncertainty with stochastic differential equations in the Stratonovich interpretation. The difference between the interpretations is mainly one of modelling, though; in fact a stochastic process x_t solves the Stratonovich equation

$$dx_t = f(x_t)\ dt + g(x_t) \circ dB_t$$

where B_t is scalar Brownian motion if and only if it solves the *equivalent* Itô equation

$$dx_t = f(x_t)\ dt + \frac{1}{2} g_x(x_t) g(x_t)\ dt + g(x_t)\ dB_t \quad .$$

See [83, p. 75, p. 32 f.] - a similar formula holds for the case of multidimensional Brownian motion. Therefore, if one has modelled a system with a Stratonovich equation, then one may afterwards do the analysis for the equivalent Itô equation.

Chapter 6

Robust performance of stochastic systems

We demonstrate that a number of performance objectives for stochastic systems correspond to stochastic dissipation requirements: stochastic $\mathcal{L}_2$ gain, $\mathcal{H}_2$ gain, probability of failure, and expected time to complete a mission. Then we consider stochastic systems subject to dissipative perturbations and show that a stochastic dissipation analysis of the nominal system can provide sufficient conditions for robust performance of the perturbed system.

6.1 Introduction

The previous chapters in this dissertation have demonstrated that dissipation theory is a very useful tool in addressing deterministic problems of robustness analysis, and that dissipation theory can be generalized to a stochastic setting. The objective of this chapter is to combine these two statements: Robust performance analysis of stochastic systems can be based on stochastic dissipation.

What motivated us to consider robust performance of stochastic systems was the specific problem of robust $\mathcal{H}_2$ performance in presence of $\mathcal{H}_\infty$

bounded perturbations. The reader may recall that we in chapter 3 gave a sufficient analysis condition for this problem, for linear systems; there we employed the deterministic interpretation of $\mathcal{H}_2$ performance which is in terms of the response to impulse inputs. This lead to the question if the same condition would also bound the $\mathcal{H}_2$ performance in the stochastic interpretation, i.e. the response to white noise. This chapter employs our notion of dissipative stochastic systems to answer this question affirmatively: For linear systems, one LMI condition on the nominal system implies robust $\mathcal{H}_2$ performance, whether the deterministic or stochastic interpretation of $\mathcal{H}_2$ performance is used. (See the note on page 142 for references to the literature on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems.)

While our original objective was to bound the variance of an error signal in presence of deterministic and stochastic uncertainty, it soon became clear that many other performance objectives could be addressed with the same framework. Essentially, if performance analysis for the unperturbed stochastic system can be cast in terms of a Lyapunov-type function on state space, then robust performance can be guaranteed by dissipation-type arguments. Particular examples of such performance measures are the risk of failure, as well as expected time to complete a mission. In this chapter we provide bounds for the risk of failure of a stochastic system in presence of deterministic dissipative perturbations; this demonstrates that it is indeed possible to merge stochastic and robust control.

The chapter is organized as follows: First, in section 6.2, we discuss performance measures for autonomous stochastic system which can be formulated in terms of dissipation. Then, in section 6.3 we add an exogenous disturbance and discuss $\mathcal{L}_2$ gain and $\mathcal{H}_2$ performance of the disturbed stochastic system. In section 6.4 we consider *finite signal-to-noise ratio* systems in the sense of Skelton and embed the associated problems in our dissipation-based approach to robust performance.

In section 6.5 we demonstrate that *robust performance* of stochastic systems subject to multi-dissipative perturbations can be guaranteed by performing dissipation analysis on the nominal system. After this general statement we present two examples: robust $\mathcal{H}_2$ performance, and robust bounds on the probability of failure. Finally section 6.6 contains a few concluding remarks.

6.2 Performance of autonomous systems

In this section we consider the autonomous stochastic system

$$\Sigma \ : \quad dx_t = f(x_t)\ dt + g(x_t)\ dB_t \tag{6.1}$$

where B_t is standard Brownian motion with respect to a filtration $\mathcal{F}_t$ on a probability space $(\Omega, \mathcal{F}, P)$. For this system we discuss two properties which may be design objectives and give sufficient conditions in terms of dissipation properties. We do not claim novelty of the conditions. Indeed, they can be found in classical literature, for instance [64]. Our contribution is simply to point out that these properties can be cast in our framework of stochastic dissipation; in particular the characterization is convex. We will employ this in a later section concerning robustness of the properties towards dissipative perturbations, thus obtaining new results.

For the convenience of the reader we include the proofs, which are all quite straightforward.

Expected time to complete a mission

Assume that the state x_t of system Σ evolves in a domain $D \subset \mathbb{X}$ and that the control mission is completed upon first exit from D. We then have the following bound on the expected time to complete the mission:

Proposition 63: Let $V : \bar{D} \to \mathbb{R}$ be a continuous storage function for Σ w.r.t. the supply rate -1; then the bound

$$E^x \tau_D \leq V(x)$$

holds for $x \in D$. △

Proof: By hypothesis the process $V(x_{t\wedge\tau_D}) + t \wedge \tau_D$ is a supermartingale and hence

$$E^x\{t \wedge \tau_D\} \leq E^x\{V(x_{t\wedge\tau_D}) + t \wedge \tau_D\} \leq V(x) \quad .$$

Since this holds for all $t \geq 0$ we conclude that τ_D is finite P^x-almost surely, and that $E^x \tau_D \leq V(x)$. ■

Risk of failure

Assume now that the boundary of D is divided into two components A and B. As before, the process is stopped upon first exit from D and the mission is denoted a *success* if A is reached, whereas exiting through B is a failure. We then have the following bound on the risk of failure:

Proposition 64: Let $V : \bar{D} \to \mathbb{R}$ be a continuous storage function for Σ w.r.t. the supply rate 0 which satisfies the additional constraint $V|_B \geq 1$; then the bound

$$P^x\{\tau_D = \tau_B\} \leq V(x)$$

holds for $x \in D$. △

Proof: By hypothesis the process $V(x_{t\wedge\tau_D})$ is a supermartingale and hence

$$P^x\{\tau_D = \tau_B\} \leq P\{\sup_{t\geq 0} V(x_{t\wedge\tau_D}) \geq 1\} \leq V(x) \quad .$$

Here the first inequality holds because $\tau_D = \tau_B$ implies $V(x_{\tau_D}) \geq 1$ and hence $\sup_{0\leq t} V(x_{t\wedge\tau_D}) \geq 1$. The last inequality is the supermartingale inequality. ■

Notice that the proposition does not claim that the probability of success is no smaller than $1 - V(x)$; this would in addition require that D is reached in finite time, P^x-almost surely. Propositions 63 and 64 may be combined to yield such a result. A related question is what the expected time to complete the mission is, conditioned on the mission being completed successfully. This is the subject of appendix A (page 151 ff.) where a new formula for this conditional expectation is derived.

6.3 Performance of disturbed systems

In this section we consider a disturbed stochastic system

$$\Sigma : \quad dx_t = f(x_t, v_t)\ dt + g(x_t, v_t)\ dB_t\ , \quad y_t = c(x_t, v_t) \qquad (6.2)$$

where v_t is the disturbance input and y_t is an output which is used in evaluation of the performance of the system. As before, B_t is Brownian motion w.r.t. a filtration $\mathcal{F}_t$. The input v_t is restricted to a set $\mathcal{V}$ of $\mathcal{F}_t$-adapted signals for which there exists a unique t-continuous solution to the dynamic equation. We assume that $\mathcal{V}$ is sufficiently large and closed under switching so that the principle of optimality holds.

Following standard notation [83], we define the backward operator associated with the controlled diffusion (6.2):

$$L^v V = V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) \quad .$$

6.3.1 Stochastic $\mathcal{L}_2$ gain

The $\mathcal{L}_2$ gain is one way of measuring the amplification, or gain, of a deterministic or stochastic system, and is a good performance measure when we adopt a worst-case view on the inputs and wish to bound their effect on the r.m.s. value of the output. A reasonable question is what makes the $\mathcal{L}_2$ norm (or the r.m.s. value) suitable as a signal norm. Here we adopt the pragmatic point of view that in many applications it is not at all clear what signal norm is suitable, and that in these situations it may be most fruitful to use the $\mathcal{L}_2$ norm since it leads to technical simplicity.

Definition 65: The stochastic $\mathcal{L}_2$ gain of the system (6.2) is denoted $||\Sigma||_\infty$ and equals the infimum of all $\gamma > 0$ such that the system is stochastically dissipative with respect to $\gamma^2|v|^2 - |y|^2$. □

Thus, we have $||\Sigma||_\infty < \gamma$ if and only if

$$E^x \int_0^\tau |y_t|^2\ dt \leq \gamma^2 E^x \int_0^\tau |v_t|^2\ dt + K(x)$$

holds for some $K : \mathbb{X} \to \mathbb{R}$ and all bounded stopping times τ and all inputs v_t in $\mathcal{V}$. In this case K must be nonnegative and may be taken to have infimum 0.

Our choice of notation suggests that $||\Sigma||_\infty$ is a norm. Indeed, it is possible to organize systems of the form (6.2) in a linear space: Fix the probability space $(\Omega, \mathcal{F}, P)$, the filtration $\mathcal{F}_t$ and the input space $\mathcal{V}$. We then view the system as a family of operators from input v_t to output z_t, parametrized by the initial condition x, and define addition and scalar multiplication of systems in the obvious way. Then the stochastic $\mathcal{L}_2$ gain $||\Sigma||_\infty$ is a semi-norm on the subspace of those systems for which it is finite.

The stochastic $\mathcal{L}_2$ gain is the one property of stochastic dissipation which has received considerable attention in the literature [30, 31].

6.3.2 $\mathcal{H}_2$ performance

Whereas it is largely agreed that the $\mathcal{L}_2$ gain is a suitable generalization of the $\mathcal{H}_\infty$ norm to nonlinear systems, it is less clear how to define the $\mathcal{H}_2$ norm of a nonlinear system defined by the stochastic differential equation (6.2). Here we suggest a new definition which is based on stochastic $\mathcal{L}_2$ gains and therefore fits into our framework of stochastic dissipation.

Since the $\mathcal{H}_2$ norm of a linear time invariant system concerns the response to a white noise input v_t, we need to modify the model (6.2) to allow for such inputs. Since we have restricted ourselves to Itô diffusions, which only allow a white noise term $d\tilde{B}/dt$ to enter *affinely* in the dynamic equation, we must assume that (6.2) is affine in v_t. Furthermore, recall [128] that a stable rational transfer function has finite $\mathcal{H}_2$ norm if and only if it is strictly causal, so we can assume that the output equation $y = c(x, v)$ is independent of v. Hence we assume that the system (6.2) has the following special form:

$$\Sigma: \quad dx_t = f(x_t)\ dt + g(x_t)\ dB_t + b(x_t)v_t\ dt, \quad y_t = c(x_t) \quad . \tag{6.3}$$

In order to define $\mathcal{H}_2$ performance of such a system Σ, we formally replace the input v_t with a white noise term $\nu_t\ d\tilde{B}/dt$. Here ν_t is a scalar *noise intensity* while $\tilde{B}_t$ is standard Brownian motion with respect to $\mathcal{F}_t$ and independent of B_t. Thus we obtain a new system, mapping the noise intensity ν_t to the output y_t:

$$\tilde{\Sigma}: \quad dx_t = f(x_t)\ dt + g(x_t)\ dB_t + b(x_t)\ \nu_t\ d\tilde{B}_t, \quad y_t = c(x_t) \tag{6.4}$$

Definition 66: The *strong $\mathcal{H}_2$ performance index* of the system (6.3) is denoted $||\Sigma||_2$ and equals the stochastic $\mathcal{L}_2$ gain of the system (6.4). $\square$

The strong $\mathcal{H}_2$ performance index is the worst-case ratio between the variance of the output y_t and the intensity of the white noise input $v_t = \nu_t\ d\tilde{B}_t/dt$. The affix *strong* is due to the feature that the intensity of the white noise input is allowed to vary, for instance as a function of the state.

Implicit in the definition is that the filtration $\mathcal{F}_t$ must be 'large enough' to allow two independent $\mathcal{F}_t$-Brownian motion processes B_t and $\tilde{B}_t$. This mathematical twist will probably cause little concern in engineering applications where we start with statistical properties of noise signals and then, usually implicitly, define the probability space accordingly.

As was the case for stochastic $\mathcal{L}_2$ gains, it is possible to organize systems Σ of the form (6.3) in a linear space such that $||\Sigma||_2$ is a seminorm on the subspace where it is finite.

We have the following partial differential inequality condition for $\mathcal{H}_2$ performance:

Proposition 67: For the system Σ defined by equation (6.3), let there exist a real number $\gamma \geq 0$ and a C^2 function $V \geq 0$ on $\mathbb{X}$ such that

$$\forall x \in \mathbb{X}: \quad V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) + \frac{1}{2}|c|^2 \leq 0\ , \quad \gamma^2 \geq \mathrm{tr}(b' V_{xx} b) \quad .$$

Then $||\Sigma||_2 \leq \gamma$. $\triangle$

Proof: We claim that V is a storage function for the system (6.4) with respect to the supply rate $\frac{1}{2}\gamma^2\nu^2 - \frac{1}{2}y^2$. The differential dissipation inequality is

$$V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) + \frac{1}{2}\nu^2\ \mathrm{tr}(b' V_{xx} b) \leq \frac{1}{2}\gamma^2\nu^2 - \frac{1}{2}|c|^2 \quad ,$$

which is seen to hold for all x and all ν if V and γ are as in the proposition. $\blacksquare$

The condition is only sufficient since storage functions need not in general be C^2. Notice that the characterization is convex in γ^2 and V. In the narrow sense linear case, i.e.

$$f(x) = Ax\ , \quad g(x) = 0\ , \quad b(x) = B\ , \quad c(x) = Cx \quad ,$$

we know from chapter 5 that we can restrict attention to quadratic storage functions, i.e. $V(x) = \frac{1}{2}x'Px$, and we recover the Lyapunov-type linear matrix inequality problem

$$P \geq 0\ , \quad PA + A'P + C'C \leq 0\ , \quad \gamma^2 \geq \mathrm{tr}(B'PB) \quad .$$

Feasibility of this problem is sufficient *and* necessary for $||\Sigma||_2 \leq \gamma$ since linear dissipative systems possess a quadratic storage function. In other words, the strong $\mathcal{H}_2$ performance index equals the standard $\mathcal{H}_2$ norm of the transfer function $C(sI - A)^{-1}B$.

It is well known [128] that for linear systems the $\mathcal{H}_2$ norm of a transfer function equals the steady-state variance of the output, when the input is white noise with unit intensity. This generalizes to nonlinear systems as follows:

Proposition 68: Let $\gamma = \|\Sigma\|_2 < \infty$ and assume that the intensity ν_t in (6.4) is constant and equal to some number $\sigma > 0$, then

$$\limsup_{T\to\infty} \frac{1}{T} E^x \int_0^T |y_t|^2 \, dt \le \sigma^2 \cdot \gamma^2 \quad .$$

If furthermore a stationary solution x_t exists such that $E\,V(x_t) < \infty$, then

$$E\,|c(x_t)|^2 \le \sigma^2 \cdot \gamma^2 \quad .$$

$\triangle$

Proof: The assumption implies that the system $\tilde{\Sigma}$ is dissipative w.r.t. $\gamma^2\nu^2 - |y|^2$. Let V be a storage function; then the dissipation inequality

$$0 \le E^x V(x_T) \le V(x) + E^x \int_0^T \gamma^2 \cdot \sigma^2 - |y_t|^2 \, dt$$

holds. This can rewritten as

$$\frac{1}{T} \int_0^T E^x |y_t|^2 \, dt \le \frac{1}{T} V(x) + \gamma^2 \cdot \sigma^2 \quad ,$$

which holds for all $T \ge 0$. Now take lim sup on both sides and notice that $\limsup_{T\to\infty} V(x)/T = 0$. The second claim follows directly from the dissipation inequality

$$E\,V(x_T) \le E\,V(x_0) + E \int_0^T \sigma^2 \cdot \gamma^2 - |c(x_t)|^2 \, dt$$

combined with the stationarity property $E\,V(x_T) = E\,V(x_0)$. $\blacksquare$

For general nonlinear systems the bounds in the proposition may be somewhat conservative since we have restricted the noise intensity ν_t to be constant.

In the deterministic case $g = 0$ the condition of proposition 67 is the existence of a Lyapunov function V for the autonomous system $\dot{x} = f(x)$ such that

$$\frac{d}{dt} V(x(t)) \le -\frac{1}{2} |c(x(t))|^2$$

and which in addition has small curvature, i.e. $\mathrm{tr}(b' V_{xx} b) \le \gamma^2$. A classical question is to what extent "nice" input-output behaviour (e.g. finite gain) implies "nice" internal behaviour (e.g. stability). In the deterministic case $g = 0$ it is possible to employ La Salle's theorem [59, p. 115, p. 440]. Hence finite strong $\mathcal{H}_2$ performance index implies asymptotic stability of the zero solution to the undisturbed system $\dot{x} = f(x)$ if: 1) the autonomous system $\dot{x} = f(x)$, $y = c(x)$ is zero-state detectable, and 2) the storage function V in proposition 67 is proper and satisfies $V^{-1}(\{0\}) = \{0\}$.

A concluding remark concerns $\mathcal{H}_2$ performance of systems (6.2) which do not have the input-affine form (6.3). In this case one needs a more general framework for stochastic differential equations than Itô diffusions, which allows a stochastic integral to be a nonlinear function of the driving martingale. Such a framework can be found in [75] but is beyond the scope of this dissertation.

6.4 FSN models

In a sequence of papers [102, 104, 100, 110, 72, 71, 103], R.E. Skelton and co-workers have introduced disturbances with *finite signal-to-noise ratio* (in short, FSN disturbances) and discussed their use for representation of uncertainty. In this section we demonstrate that FSN disturbances can, too, be represented in the framework of stochastic dissipation.

FSN disturbances are white noise signals with intensities which are not fixed *a priori* but grow with the variance of specified signals in the closed loop. As argued in [102], this is a reasonable model of round-off errors in finite word-length computations with floating point, as well as of turbulence forces around air foils.

To be more specific, consider the linear system

$$\dot{x} = Ax + Gw \;, \quad y = Cx \tag{6.5}$$

where w is an FSN disturbance: i.e., a scalar white noise signal with intensity $\sigma_0^2 + \sigma_1^2 E(y'y)$. Here σ_0 and σ_1 are specified constants; σ_1 is called the *noise-to-signal ratio*. Also other terms such as controls may appear in the expressions for $\dot{x}$ and y but are irrelevant to the present discussion. The model can be generalized to allow for vector disturbances w in several ways.

The model (6.5) is well suited for steady-state analysis: A unique invariant distribution for x exists if and only if there exists a unique non-negative solution P to the generalized Lyapunov equation

$$PA' + AP + GG'(\sigma_0^2 + \sigma_1^2 \cdot \mathrm{tr}(CPC')) = 0 \quad . \tag{6.6}$$

In this case, this unique invariant distribution for x is $N(0, P)$, i.e. in steady state x is zero mean, has covariance P and is Gaussian. However, the model (6.5) does not fully describe the process x. For instance, assume that we have observed x up to some time t, what is then the conditional distribution of w? Such questions are important if one wishes to study transient behaviour.

The objective of this section is twofold: First, we wish to generalize FSN models to nonlinear and non-stationary systems. Second, we wish to formulate FSN models such that they can be combined with our dissipation-based framework for robustness. We believe that the following model fulfills both objectives:

$$dx_t = f(x_t)\ dt + g(x_t)\left(\sigma_0 \ \ dB_t + \sigma_1\zeta_t\ d\tilde{B}_t\right) \ , \quad y_t = c(x_t) \quad . \tag{6.7}$$

Here ζ_t is the scalar output of an unknown deterministic system Δ which has $\mathcal{L}_2$ gain less than or equal to 1, and the input of which is y_t. Furthermore B_t and $\tilde{B}_t$ are independent standard Brownian motion.

In order to see that this is indeed a generalization of the model (6.5), set $f(x) = Ax$, $g(x) = G$, $c(x) = Cx$, and assume that steady-state has been reached. Assume furthermore that Δ is a *worst-case* perturbation so that the root mean square (r.m.s.) of ζ_t equals that of y_t. Then it is fairly easy to see that x in steady state must have zero mean and variance P where P solves the generalized Lyapunov equation (6.6), which implies that the models (6.5) and (6.7) lead to the same steady-state mean and variance. However, our suggested model (6.7) need not lead to Gaussian distributions in steady state - this will depend on the particular system Δ.

If one wishes to simulate an FSN system, one will obviously have to choose a particular perturbation Δ. Two systems with $\mathcal{L}_2$ gain equal to one are of special interest:

$$\Delta: \quad \zeta(t) = \ y(t) \qquad \text{(for } y \text{ scalar)}$$

$$\Delta: \quad \zeta(t) = \sqrt{\int_0^\infty \omega \exp(-\omega\tau)|y(t-\tau)|^2\ d\tau}$$

The first of these two specific perturbations Δ is that of *multiplicative noise*, c.f. e.g. [31] and the references therein. In certain analysis problems for linear FSN systems this perturbation is *worst case*. The second form of Δ illustrates that ζ may be thought of as an r.m.s. estimator for y.

When we analyse FSN models, we take the the perturbation Δ to be an unknown state-space system with $\mathcal{L}_2$ gain less than or equal to one, and we adopt a worst-case view on this class of perturbations. In particular applications, one may possess additional knowledge regarding Δ, for instance concerning time constants.

We have thus demonstrated that FSN models can be embedded in our general framework for uncertain systems; viz. a nominal system described by a stochastic differential equation, subject to an unknown perturbation which possesses a number of specified dissipation properties.

6.5 Performance of perturbed systems

In this section we consider the interconnection of a nominal stochastic system Σ and a multi-dissipative deterministic perturbation Δ; see figure 6.1. Our objective is to provide conditions on the nominal system Σ under which the interconnection dissipates a given supply rate r for any multi-dissipative perturbation Δ. This is a fairly general problem formulation; later we consider specific applications such as robust $\mathcal{H}_2$ performance in presence of $\mathcal{H}_\infty$ bounded perturbations.

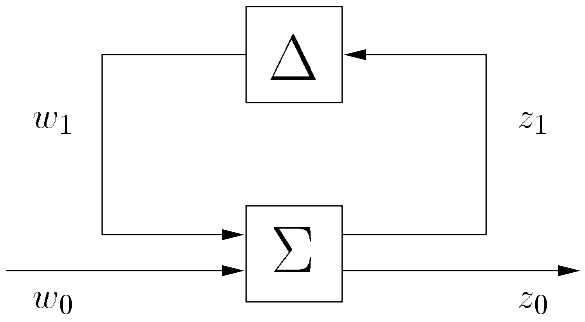

Figure 6.1: Setup for robust performance analysis.

The nominal system Σ is described by a stochastic differential equation

$$\Sigma : \quad \begin{array}{rcl} dx_t & = & f(x_t, w_t, v_t)\ dt + g(x_t, w_t, v_t)\ dB_t\ , \\ y_t & = & c(x_t, w_t, v_t)\ , \\ z_t & = & h(x_t, w_t, v_t)\ , \end{array} \tag{6.8}$$

whereas the unknown perturbation Δ, mapping z to w, is known to dissipate p given supply rates $-r_i$, $i = 1, \ldots, p$. We let ξ denote the state of the perturbation Δ and let $W(\xi, -r_i)$ denote a storage function for Δ w.r.t. $-r_i$. The backwards operator corresponding to (6.8) is

$$L^{w,v} V(x) = V_x f + \frac{1}{2} \mathrm{tr}(g' V_{xx} g)$$

for $V \in C^2(\mathbb{X}, \mathbb{R})$; here the right hand side is evaluated at x, w, v.

We omit details concerning well-posedness of the interconnection; i.e. we assume that unique t-continuous solutions x_t, ξ_t exist for any $\mathcal{F}_t$-adapted input v_t in a sufficiently large class of inputs.

The vehicle of our analysis of the interconnection (Σ, Δ) is an extended system derived from the nominal system Σ and independent of Δ: Define the system $\bar{\Sigma}$ by appending to (6.8) the dynamic equation

$$d\beta_t^i = -r_i\ dt \quad . \tag{6.9}$$

Thus $\bar{\Sigma}$ has states x_t and $\beta_t = (\beta_t^1, \ldots, \beta_t^p)$, inputs w_t and v_t, and outputs z_t and y_t. The backwards operator associated with $\bar{\Sigma}$ is

$$M^{w,v} U(x, \beta) = U_x f - \sum_{i=1}^{p} U_{\beta_i} r_i + \frac{1}{2} \mathrm{tr}(g' U_{xx} g)$$

for $U \in C^2(\mathbb{X} \times \mathbb{R}^p, \mathbb{R})$; here the right hand side is evaluated at x, β, w, v.

Definition 69: We say that $\bar{\Sigma}$ is *regionally dissipative* on $D \subset \mathbb{X} \times \mathbb{R}^p$ w.r.t. the supply rate r if there exists a function $U(x, \beta)$ which is non-negative on D and such that the dissipation inequality

$$E^{x,\beta} U(x_\tau, \beta_\tau) \le U(x, \beta) + E^{x,\beta} \int_0^\tau r\ dt \tag{6.10}$$

holds for all $(x, \beta) \in D$, all $\mathcal{F}_t$-adapted inputs v_t, w_t and all bounded stopping times τ such that

$$\tau \le \tau_D := \inf\{t \ge 0\ :\ (x_t, \beta_t) \notin D\} \tag{6.11}$$

holds. □

Regionally dissipative systems are not directly covered by our definition of stochastic dissipation (page 105); nevertheless it is straightforward to verify that they possess many properties similar to those of dissipative systems. Let us only state the partial differential inequality condition:

Proposition 70: Let $D \subset \mathbb{X} \times \mathbb{R}_+^p$ be open and let $U \in C^2(D, \bar{\mathbb{R}}_+)$. The following are equivalent:

1. U is a regional storage function for $\bar{\Sigma}$ on D w.r.t. the supply rate r.
2. U satisfies the partial differential inequality

$$\sup_{w,v} M^{w,v} U(x, \beta) - r \le 0 \tag{6.12}$$

on D.

△

Proof: The proof is merely a repetition of the proof of proposition 51 on page 106 and omitted. ■

We can now state our main result which is a sufficient condition for the interconnection (Σ, Δ) to dissipate r.

Theorem 71: Assume that $\bar{\Sigma}$ is regionally dissipative on $\mathbb{X} \times \mathbb{R}_+^p$ w.r.t. r with $U(x, \beta)$ a corresponding regional storage function. Then the interconnection (Σ, Δ) dissipates r; an upper bound on the available storage is

$$U(x, \beta)$$

provided that $\beta^i > W(\xi, -r_i)$. □

The idea behind the theorem is that the appended states β_t^i of $\bar{\Sigma}$ bound the storage $W(\xi_t, -r_i)$ in the perturbation. This technique has, to our knowledge, not been used before in the literature; even in a deterministic context.

Proof: Consider the response x_t, ξ_t of the interconnection (Σ, Δ) to an $\mathcal{F}_t$-adapted input v_t under the initial conditions x and ξ. Let $\beta^i > W(\xi, -r_i)$. We aim to show that

$$E^{x,\xi} \int_0^\tau -r\ dt \le U(x, \beta) \tag{6.13}$$

holds for any bounded stopping time τ.

First, notice that the output w_t of Δ is $\mathcal{F}_t$-adapted since Δ is deterministic. Let $\bar{x}_t$, β_t be the response of $\bar{\Sigma}$ to the inputs v_t and w_t and the initial conditions x and β. Then clearly $x_t = \bar{x}_t$ by uniqueness; the processes x_t and $\bar{x}_t$ solve the same stochastic differential equation (6.8) with the same initial condition.

Next, the dissipation inequalities for Δ are

$$0 \le W(\xi_t, -r_i) \le \int_0^t -r_i \; ds + W(\xi, -r_i)$$

and hold for *any* sample trajectory and any $t \ge 0$. This implies that

$$0 \le \delta_t^i \le \beta_t^i - \beta^i + W(\xi, -r_i) < \beta_t^i \quad .$$

Finally, let τ be a bounded stopping time. Since $\beta_t^i > 0$ for any $t \ge 0$, the regional dissipativity of $\bar{\Sigma}$ implies that

$$E^{x,\beta} \int_0^\tau -r \; dt \le U(x, \beta) - E^{x,\beta} U(x_\tau, \beta_\tau) \le U(x, \beta)$$

which completes the proof. ∎

Linear combinations of supply rates

Theorem 71 generalizes the conditions of chapter 3 where we required the nominal system Σ to dissipate a linear combination of the supply rates r, r_i. We may recover this type of results (in a stochastic context) by imposing a specific structure on U:

Corollary 72: Assume that there exists non-negative weights d_i such that Σ dissipates the supply rate

$$r + \sum_{i=0}^{p} d_i r_i$$

then the interconnection (Σ, Δ) dissipates r. □

Proof: In the theorem, take $U(x, \beta) = V(x) + \sum_i d_i \beta_i$ where V is a storage function of Σ w.r.t. the supply rate $r + \sum_{i=0}^{p} d_i r_i$. Let w_t, v_t be $\mathcal{F}_t$-adapted inputs to Σ and $\bar{\Sigma}$ and let τ be bounded; we then have

$$\begin{aligned} E^{x,\beta} U(x_\tau, \beta_\tau) &= E^x V(x_\tau) + E^{x,\beta} \sum_{i=1}^{p} d_i \beta_\tau^i \\ &\le V(x) + E^x \int_0^\tau r + \sum_{i=1}^{p} d_i r_i \; dt \\ &\quad + E^{x,\beta} \sum_{i=1}^{p} d_i (\beta^i - \int_0^\tau r_i \; dt) \\ &= E^x \int_0^\tau r \; dt + U(x, \beta) \end{aligned}$$

which implies that the sufficient condition of theorem 71 is satisfied. Notice that $U(x, W(\xi, -r_i))$ is in this case in fact a storage function for the interconnection (Σ, Δ). ∎

Conservatism of the condition

Since theorem 71 provides a sufficient condition, but not a necessary one, the question is how conservative the condition is. Before we discuss this issue we emphasize that the condition is less conservative than those of chapter 3; this is demonstrated by corollary 72. In fact the condition of theorem 71 is not very conservative.

First, the theorem does not only guarantee that the interconnection (Σ, Δ) dissipates r but also that there exists a bound on the available storage which depends only on $W(\xi, -r_i)$, and not on the actual perturbation Δ and its initial condition ξ. This may be conservative if all we care about is that the interconnection is dissipative. On the other hand, in most applications it does not suffice to know that a bound exists for the available storage of (Σ, Δ); we also want to know what this bound is. Since the initial storage in Δ may very well be the one quantity we can bound reliably, it is appealing that this is exactly what we need to bound the available storage of (Σ, Δ).

Another way conservatism is introduced in the theorem is that the dissipation inequality (6.10) holds for all $\mathcal{F}_t$-adapted inputs w_t. Notice that a deterministic perturbation Δ must necessarily produce an output w_t which is adapted to the sub-filtration generated by z_t. In other words, the theorem is conservative in that the bound (6.13) holds also for perturbations which have access to complete information about the system Σ. This conservatism may even be desirable in applications where Δ is physically integrated in the total control system; for instance if Δ represents parasitic high-frequency dynamics. Then it would be hazardous to let a design depend on Δ not exchanging information with its environment.

A similar discussion concerns the situation where the perturbation Δ is composed of a large number of independent blocks in parallel, i.e. $w^i = \Delta^i z^i$. It appears to be difficult to make use of the fact that multiple perturbations really must solve decentralized control problems in order to make the dissipation inequality fail. In short, we restrict the energy and other resources available to Δ; not the information.

Refining the storage bounds $\beta_t^i > W(\xi_t, -r_i)$

The idea in theorem 71 is that we keep track of how much storage is present in the perturbation Δ through the bounds

$$\beta_t^i > W(\xi_t, -r_i) \quad .$$

The dynamic equation $d\beta_t^i = -r_i \; dt$ simply states that if we supply a quantity to Δ, then the storage in Δ may increase with this quantity but no more.

In some applications it may be essential to incorporate additional knowledge about Δ such as time constants. For instance, consider a welding robot which first moves the arm into correct position with large and fast movements after which the welding process begins and the welding seam is to be followed slowly and accurately. The perturbation Δ is parasitic high-frequency dynamics in the robot arm; the storage in Δ is mechanical energy. During the initial rough placement of the robot arm it is likely that large amounts of energy is supplied to the perturbation. It is then important for the analysis that this energy cannot be hidden in Δ and then released much later, during the fine movements of the actual welding process. In such a situation one may replace the dynamic equation for β_t^i with

$$d\beta_t^i = (-\frac{1}{T_\Delta}\beta_t^i - r_i) \; dt$$

where T_Δ is the time constant of the perturbation. Of course, also other forms of decay can be used, for instance if physical reasoning gives bounds

to the storage which Δ is capable to keep. In general, these issues are important if some phases of the system operation are more critical or sensitive than others.

The idea of bounding the storage in the perturbation has applications far beyond the robustness analysis which we concentrate on here. For instance, a supervisory system may keep track on-line of the storage in the perturbation using the dynamic equation of β_t^i as well as on-line measurements from the system. A large storage may provoke an alarm, or pause the control mission until the storage in the perturbation decreases to an acceptable level. For the welding robot above, this means to stop welding until we are confident that parasitic oscillations in the arm have died out. On the other hand, if the bound β_t^i ever goes negative then it can be concluded that the model is inconsistent with the measurements which may trigger a change of control strategy. The reference [88] describes an approach to adaptive $\mathcal{H}_\infty$ control based on a finite number of models and this type of model validation.

6.5.1 Guaranteed $\mathcal{H}_2$ performance

Consider now the block diagram in figure 6.2 where the system Σ has inputs w_t, σ_t and v_t and is given by the model

$$\Sigma : \quad dx_t = f(x_t, w_t) \; dt + \sigma_t \; g(x_t) \; dB_t + v_t \; b(x_t) \; dt \tag{6.14}$$

with outputs $y_t = c(x_t)$, $\zeta_t = \eta(x_t)$, and $z_t = h(x_t)$. We make the following assumptions about the perturbations Δ and Δ_F:

Δ is passive and small $\mathcal{L}_2$-gain, i.e. dissipative w.r.t. $-r_1 = \langle w, z\rangle$ and $-r_2 = |z|^2 - |w|^2$. This could for instance represent unmodelled parasitic dynamics.

Δ_F is small $\mathcal{L}_2$-gain, i.e. dissipative w.r.t. $-r_3 = |\zeta|^2 - |\sigma|^2$. This implies that $\sigma_t \; dB_t/dt$ is a white noise signal which grows in intensity with the variance of ζ_t, i.e. a *finite signal-to-noise ratio disturbance*.

To evaluate the strong $\mathcal{H}_2$ performance index of the total system, we follow our definition 66 and replace the input v_t in (6.14) with a white noise term $\nu_t \; dW_t/dt$, thus obtaining

$$\tilde{\Sigma} : \quad dx_t = f(x_t, w_t) \; dt + \sigma_t \; g(x_t) \; dB_t + \nu_t \; b(x_t) \; dW_t \quad . \tag{6.15}$$

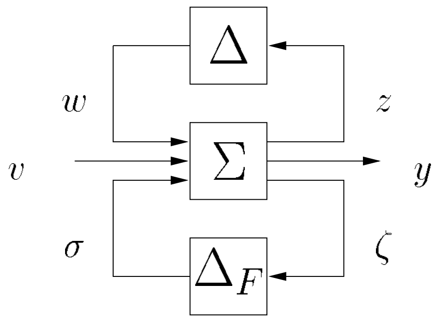

Figure 6.2: Nominal system and perturbations

Assume now that $\tilde{\Sigma}$ is stochastically dissipative w.r.t. $\gamma^2|\nu|^2 - |y|^2 + \sum_{i=1}^3 d_i r_i$ for some $\gamma \geq 0$, $d_i \geq 0$ and that V is a corresponding storage function, then it follows from corollary 72 that the overall interconnection is dissipative w.r.t. $\gamma^2|\nu|^2 - |y|^2$; a storage function is $V + \sum_i d_i W_i$. Hence, an upper bound on the square of the strong $\mathcal{H}_2$ performance index is

$$\min_{\gamma^2, d_i, V} \gamma^2 \quad \text{s.t.} \quad V \text{ a storage function for (6.15) w.r.t. } \gamma^2|\nu|^2 - |y|^2 + \sum_{i=1}^{3} d_i r_i$$

where $\gamma \geq 0$ and $d_i \geq 0$. This infinite-dimensional optimization problem is convex according to proposition 53; if the state x has low dimension it may be solved be restricting V to a finite-dimensional subspace as outlined on page 117.

If the right hand side of the governing equation (6.14) is linear in (x, w, v, σ) then V can be taken to be quadratic and the optimization problem reduces to a linear matrix inequality problem:

Theorem 73: Let the system Σ be given by the linear SDE

$$\Sigma \ : \ dx_t = (Ax_t + \Phi w_t + Bv_t) \ dt + \sigma_t \ G \ dB_t$$

and the output equations $z_t = Hx_t$, $y_t = Cx_t$, $\zeta_t = Jx_t$, and let $w = \Delta z$ and $\nu = \Delta\zeta$ where Δ and Δ_F are as above. Then an upper bound on the square of the strong $\mathcal{H}_2$ performance index of the interconnection is

$$\min_{P, d_1, d_2} \text{tr } B'PB \quad \text{s.t.} \quad P \geq 0, \ d_i \geq 0, \ \begin{bmatrix} Y & P\Phi + d_1 H' \\ \Phi' P + d_1 H & -d_2 I \end{bmatrix} \leq 0$$

where Y is shorthand for $Y = PA + A'P + C'C + d_2 H'H + J'J \text{ tr } G'PG$. □

Proof: The proof is merely a verification that, given feasible P and d_1, d_2, the quadratic form $V(x) = x'Px$ is a storage function of system

$$dx_t = (Ax_t + \Phi w_t) \ dt + \nu_t \ B \ dW_t + \sigma_t \ G \ dB_t$$

with respect to the supply rate $\gamma^2|\nu|^2 - |y|^2 + \sum_{i=1}^3 d_i r_i$ with $\gamma^2 = \text{tr } B'PB$ and $d_3 = \text{tr } G'PG$. ■

This upper bound can be computed with standard software for linear matrix inequalities such as [38, 32]. Notice that if one removes the FSN disturbance $\sigma_t \ g(x_t) \ dB_t/dt$ in (6.14) and applies the condition in theorem 27 on page 61 for robust $\mathcal{H}_2$ performance in the deterministic sense, then one recovers the condition of theorem 73. On other words, if one is after sufficient conditions for robust $\mathcal{H}_2$ performance of linear systems, then it is inessential if one uses the stochastic or the deterministic interpretation of $\mathcal{H}_2$ performance.

6.5.2 Robust estimates on the risk of failure

Consider a system

$$\Sigma \ : \ dx_t = f(x_t, w_t) \ dt + g(x_t, w_t) \ dB_t \ , \quad z_t = h(x_t) \tag{6.16}$$

connected in feedback with a deterministic perturbation $\Delta : z \to w$ which dissipates the p supply rates $-r_1, \ldots, -r_p$. Let the initial condition x be in an open domain $D \subset \mathbb{X}$, let the boundary ∂D be divided into two disjoint sets A and B; corresponding to success and failure, respectively.

As before, we let $\bar{\Sigma}$ denote the system Σ appended with the states β_t^i with $d\beta_t^i = -r_i \ dt$.

Theorem 74: Assume that $\bar{\Sigma}$ is regionally dissipative on $D \times \mathbb{R}_+^p$ w.r.t. the supply rate 0 with a regional storage function $U(x, \beta)$ which is continuous on $\bar{D} \times \mathbb{R}_+^p$ and such that $U(x, \beta) \geq 1$ whenever $x \in B$ and $\beta^i \geq 0$. Then we have the following bound on the risk of failure

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq U(x, \beta)$$

where $\beta^i = W(\xi, -r_i)$. □

Proof: Let $\beta^i > W(\xi, -r_i)$ and let x_t, β_t be the trajectories of $\bar{\Sigma}$ when connected in feed-back with Δ, corresponding to the initial conditions x, ξ and β. We claim that the process $U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D})$ is a continuous supermartingale. Continuity is clear since x_t and β_t are continuous processes and U is a continuous function. To see that the process is a supermartingale, notice that $\beta_t^i > W(\xi_t, -r_i) \geq 0$, and hence regional dissipativity w.r.t. the supply rate 0 yields

$$E^{x,\xi} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \leq U(x, \beta) \quad .$$

This allows us to pose the probability bound

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq P^{x,\beta}\{\sup_{0\leq t} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \geq 1\} \leq U(x,\beta) \quad .$$

Here, the first inequality holds because $x_{\tau_D} \in B$ implies that $U(x_{\tau_D}, \beta_{\tau_D}) \geq 1$ and hence $\sup_{0\leq t} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \geq 1$. The second inequality is the supermartingale inequality.

We have thus shown that $P^{x,\xi}\{x_{\tau_D} \in B\} \leq U(x,\beta)$ for any β such that $\beta^i > W(\xi, -r_i)$. Now let $\beta^i \to W(\xi, -r_i)$ from above and use continuity of U to see that the same bound holds with $\beta^i = W(\xi, -r_i)$. ■

A similar conclusion is obtained if we follow corollary 72 and replace the hypothesis with Σ dissipating $\sum_i d_i r_i$ for non-negative weights d_i, with a continuous storage function V such that $V|_B \geq 1$. However, in this case the resulting bound is

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq V(x) + \sum_i d_i W(\xi, -r_i)$$

which is seen to be quite conservative for large amounts of initial storage in the perturbation Δ; in fact the upper bound may then become $P^{x,\xi}\{x_{\tau_D} \in B\} \leq 1$ which is not very informative. In this situation theorem 71 is of much more use; at least for large amounts of initial storage in the perturbation. In other words, it may well be very conservative to consider only regional storage functions $U(x,\beta)$ which are affine in β.

6.6 Conclusion

This chapter has demonstrated that problems of robust performance of stochastic systems can be addressed with the notion of stochastic dissipation. The three steps in this procedure are:

1. Model the physical system as an interconnection of a nominal system Σ and a perturbation Δ, where Δ is dissipative w.r.t. the supply rates $-r_i$, $i = 1, \ldots, p$.

2. Formulate the performance property as one of stochastic dissipation, i.e. find the supply rate r such that the overall system has satisfactory performance iff it dissipates r.

3. Perform dissipation analysis on Σ using theorem 71 or corollary 72, i.e. investigate if $\bar{\Sigma}$ dissipates r regionally, or if Σ dissipates $r+\sum_i d_i r_i$ for non-negative weights d_i.

Regarding the first item, the dissipation properties of Δ will typically be the same as in a deterministic analysis, such as passivity or small gain. We have also demonstrated that Skelton's finite signal-to-nose ratio (FSN) models can be incorporated in this framework.

Regarding the second item, we have shown that stochastic $\mathcal{L}_2$ gain, $\mathcal{H}_2$ performance, risk of failure and expected time to complete a mission are examples of performance objectives which can be stated in terms of stochastic dissipation. While it is hardly surprising that the stochastic $\mathcal{L}_2$ gain is related to dissipation, it is an innovation that $\mathcal{H}_2$ performance is expressed in this framework. We believe that nonlinear $\mathcal{H}_2$ control, both nominal and robust, is a fruitful field of future research. The two last performance measures, risk of failure and expected time to complete a mission, are well studied in the classical literature on stochastic analysis and control, but it is a novelty that they can be embedded in the framework of dissipation and thus subjected to a robustness analysis.

Regarding the last item, the idea of searching through convex conic combinations of supply rates was also employed (in a deterministic context) in chapter 3 and in the recent reference [126], but it is a new observation that this idea is a special case of regional dissipation analysis of the extended system $\bar{\Sigma}$; i.e. that corollary 72 follows from theorem 71.

The practical applicability of our suggested framework depends on two factors: First, we need numerical methods for performing (regional) dissipation analysis on general nonlinear systems - here it would be interesting to develop the LMI based procedure suggested on page 117 and apply it to some benchmark problems. Second, recognizing that these numerical methods will not be applicable to systems with high-dimensional state spaces due to the curse of dimensionality, we need analytical procedures for simplifying the dissipation analysis using information about the structure of the system. Modularity is one such procedure; time-scale separation would be another interesting issue to investigate.

6.7 Notes and references

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems

The literature contains several different statements of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ analysis and control problems, [26, 60, 68, 69, 86, 98, 106, 129, 132]. Much of this work concerns posing an $\mathcal{H}_2$ bound on one closed loop transfer function and an $\mathcal{H}_\infty$ bound on another. Problems of robust $\mathcal{H}_2$ performance of a linear system in presence of one $\mathcal{H}_\infty$ bounded perturbation are treated in [106, 86]. The setting there is much alike the one used in section 6.5.1; however the object of analysis in these references is a family of Riccati equations rather than a linear matrix inequality. The parameter in this family corresponds to our weight d_1. The final numerical strategy is then to search over this weight, solving a Riccati equation for each d_1. This approach is difficult with more than one dissipation property of the perturbation, since it is not clear how the solution of the Riccati equation depends on the d-weights. We have in [113] presented a numerical example with two dissipation properties; for this example a convexity property makes numerical optimization over the d-weights feasible.

Stability of FSN systems

The simplest FSN model, according to our suggested definition, is

$$dx_t = f(x_t)\ dt + g(x_t)\zeta_t\ dB_t\ , \quad y_t = c(x_t)$$

where $\zeta_t = \Delta y_t$; here Δ is a deterministic system with $\mathcal{L}_2$ gain less than or equal to one. This corresponds to (6.7) where the signal-to-noise ratio σ_1 is 1, and $\sigma_0 = 0$. A sufficient condition for this system to be stable is that the stochastic $\mathcal{L}_2$ gain from ζ_t to y_t is less than one; this is equivalent to the system mapping v_t to y_t given by

$$dx_t = f(x_t)\ dt + g(x_t)v_t\ dt\ , \quad y_t = c(x_t)$$

having strong $\mathcal{H}_2$ performance index less than 1. This is a *small gain* type result for nonlinear FSN systems.

Earlier joint work with R.E. Skelton [110], for linear FSN systems, concluded that this condition was sufficient *and* necessary. Furthermore, for the situation with *several* FSN disturbances, a necessary and sufficient condition was given in terms of the spectral radius of a certain matrix, the elements of which were obtained by $\mathcal{H}_2$ analysis on the nominal system. It is in fact possible to give a similar *sufficient* condition for stability of nonlinear FSN systems with several FSN disturbances, employing corollary 72. This result will be reported elsewhere.

Chapter 7

Conclusion

We have in this dissertation contributed to the mathematical theory of robust performance of control systems in presence of parametric uncertainty, dynamic perturbations, and deterministic or stochastic exogenous disturbances.

There are four threads in our work. The first is the opinion that control theory should employ notions which have some general validity and not only, for instance, make sense in a deterministic linear setting. We believe that our dissipation based framework for robust performance of stochastic systems fulfills this requirement.

The second thread is the opinion that control theory should maintain a close connection to physics. This is partly because many techniques from physics, such as Lyapunov stability, has proven to be valuable to control theorists, but also because a sound knowledge of the physics in a control system will assist the control engineer in posing the right mathematical problems.

Thirdly, we consider the uncertainty associated with a nominal mathematical model to be equally important as the nominal model itself. The representation of uncertainty determines the strategy for analysis and design, and the more detailed the information about the uncertainty, the sharper conclusions. Both the simultaneous $\mathcal{H}_\infty$ controller of chapter 4 and the robust performance analysis of chapter 6 uses explicit quantitative evaluation of the uncertainty, in terms of the residuals and the storage of the perturbations.

Lastly, we believe that tools for analysis of control systems are as important as tools for synthesis. Good analysis tools, which for instance could be based on dissipation analysis on a closed loop system, can be of great practical value, not only for the theorist but also for the practicing engineer. For instance an inspection of storage functions may conclude that a heuristic controller, although not optimal, solves the control job nicely, or it may identify a weakness in the design of the *plant.*

In the remainder of this chapter we briefly summarize the precise nature of our contributions, and point out a number of issues which deserve further attention.

7.1 Summary of contributions

The purpose of this section is to provide a concentrated overview of the results which were obtained during the Ph.D. study and reported in this thesis.

The introductory chapter 1 does not present new results, although the observation that LMIs can be used to compute storage functions for nonlinear but input affine-quadratic systems seems to be new.

Chapter 2 presents fundamental properties of deterministic systems which are dissipative w.r.t. several supply rates. The convex conicity of the set of dissipated supply rates is mentioned in passing in [45]; this simple property is what enables the robustness analysis of the succeeding chapter. New results are that the set is also closed and that the available storage is a continuous function on this set. These properties are important for a numerical analysis and contribute to the general understanding of multi-dissipative systems.

The contribution of chapter 3 is to demonstrate that analysis of control systems can be done by explicit consideration of the multiple dissipation properties of unknown system components. It is fair to say that this idea is also present in approach of Integral Quadratic Constraints, but several differences exist between this framework and the one of multi-dissipation as explained in section 3.1. The chapter also contains several more technical

contributions which can be seen as exercises in Lyapunov techniques - in this type of work, the devil is in the details. The results for linear-quadratic systems are obtained using standard methods for linear matrix inequalities. The importance of these results is to demonstrate that problems with such mixed uncertainty models lead to convex optimization problems, namely LMIs.

Chapter 4 contributes to the theory of adaptive $\mathcal{H}_\infty$ control by pointing out that certainty equivalence based minimax controllers for this problem is not the generic situation. Although the characterization of the minimax controller is done with existing ideas, viz. the information state machinery, the literature contains few applications of this machinery, and the details are by no means trivial. One such detail is the characterization of the value function as the viscosity solution to the HJI-PDE. In a given application it will be a cumbersome affair to construct the minimax controller, but it is quite straightforward to synthesize the heuristic certainty equivalence controller, and this design may have direct practical applicability.

Chapter 5 contains a generalization of dissipation theory to stochastic systems. In the existing literature, dissipation techniques have only been used to perform analysis of stochastic systems in special cases; it appears to be a new observation that the framework is applicable and operational in general. The results of the chapter essentially say that many of the attractive features of deterministic dissipative systems apply to stochastic dissipative systems as well; these are the inherent convexity, the rôle of the available storage, the closedness under interconnections, and the implications for stability. The strictly positive real lemma for wide-sense linear stochastic systems is new; passivity of stochastic systems has to our knowledge not been investigated previously.

Chapter 6 constructs a framework for robustness of stochastic systems, based on the theory of stochastic dissipation. A minor contribution is the observation that stochastic performance measures such as the risk of failure can be formulated in terms of dissipation. It is more innovative that the same applies to $\mathcal{H}_2$ performance and finite signal-to-noise ratio (FSN) models. The idea of expanding the system with extra states, which keep track of the storage in the perturbation, is new. This idea leads to quite sharp sufficient conditions for robust performance; for general nonlinear systems these conditions are more natural than the multiplier-based approach of chapter 3. The idea may also have further applicability in other fields of control theory such as supervision and model validation.

7.2 Perspectives and future works

As is so often the case, each of the answers in this dissertation leads to several new questions. Many of the results could be refined or generalized; the *notes and references* ending each chapter contains such detailed suggestions for future works. At this point we take a step back and outline some fields of research which we believe to be fertile.

The problem of adaptive $\mathcal{H}_\infty$ control remains largely open. As stated in chapter 4, we cannot expect the minimax controller to be based on certainty equivalence or finite dimensional (when there is more than a finite number of possible parameter values). In this situation there is a great need for clever heuristics and sub-optimal strategies as well as for studies of special situations, and although much work has been done in this direction, there are many questions that remain unaddressed. A fundamental question is if the problem formulation itself is a sign of prudence or paranoia. In other words, should we impose some further constraints on those disturbances for which the dissipation inequality must hold, or is it reasonable to anticipate disturbances which in some clever way attempt to confuse the control system?

We have, in the notes at the end of chapter 5, mentioned the possibility of extended the theory of stochastic dissipation to a more general class of stochastic differential equations than Itô diffusions. A related interesting project would be to extend the theory of stochastic dissipation to infinite-dimensional systems, i.e. systems given by stochastic partial differential equations. Initial results in this direction are probably obtained quite easily, following [124] where many results hold for infinite dimensional systems, but we expect it to be quite complicated to obtain more explicit results. A good starting point for such a project would be the corresponding deterministic problem, see [61] and the references therein.

As we have already mentioned on several occasions, numerical methods for analysis and control of nonlinear systems remains the hurdle for the practical applicability of the theory, and is a natural subject of future investigations.

After the robustness analysis results of chapter 6, an obvious next step is to develop a theory of control for stochastic dissipation. The objective of such a theory is to provide techniques for finding a control law, a storage function, and possibly also a supply rate in a given set, which together satisfy

the dissipation inequality. In principle, this can be done by value-policy iteration but we expect that much more explicit results can be obtained, at least if some generality is sacrificed.

A special case of such a theory is *nonlinear $\mathcal{H}_2$ control* building on the definition of strong $\mathcal{H}_2$ performance index of chapter 6. The term nonlinear $\mathcal{H}_2$ control is most often used in the deterministic meaning, where the cost is evaluated from the response to initial conditions, and is therefore unable to conclude on the response to white noise. Similarly, stochastic nonlinear optimal control is most often used with *fixed* noise intensities, and does therefore not provide information about the response to other noise intensities. In some applications it is quite sensible to take a worst-case view on the noise intensity (as in our definition of strong $\mathcal{H}_2$ performance). It also embeds nicely in a dissipation-based robustness framework - notice that fixing the noise intensity (or just bounding it way from zero) leads to supply rates which are not regular and thus weakens the dissipation theory. In short, we believe dissipation-based nonlinear $\mathcal{H}_2$ control to be a promising field.

Appendix A

Conditional Expectations of First Passage Times

We consider an Itô diffusion evolving on a domain in Euclidean space, the boundary of which is divided into two components, A and B. We then ask the question: What is the expected time to pass before the set A is reached, conditioned on A being reached before B?

We derive a partial differential equation which governs this conditionally expected first passage time, seen as a function of the initial state. We also provide a generalization which involves other functionals than the first time of exit, and we show how a partial differential inequality can be useful for establishing bounds.

A classical question concerning Itô diffusions evolving in Euclidean spaces is: If the diffusion starts at a point x in some open set Ω, what is the expected time $E^x \tau_{\partial\Omega}$ to pass before it reaches the boundary $\partial\Omega$? It is well known that under suitable technical assumptions this expected first passage time, seen as a function of the initial state x, is the unique solution to the second order semi-elliptic partial differential equation

$$L\phi = -1, \quad \phi|_{\partial\Omega} = 0$$

Here, L is the backward differential operator associated with the diffusion - see below for precise definitions and statements.

A related question is: If we divide the boundary $\partial\Omega$ into two disjoint components A and $B = \partial\Omega \setminus A$, what is the probability $P^x\{\tau_{\partial\Omega} = \tau_A\}$ that the process hits A before B? This probability is - again, under suitable technical assumptions - the unique solution to the equation

$$L\psi = 0, \quad \psi|_A = 1, \quad \psi|_B = 0 \quad .$$

One application of these results is performance analysis of a stochastic control system: The control mission is completed upon passage of the boundary $\partial\Omega$; successfully if the boundary is reached at a point in A whereas reaching B before A would be a failure. For instance, the mission could be docking of a ship or a spacecraft. The primary performance measure for this application may be the probability of success, i.e. the function ψ, whereas a secondary performance measure may be the time it takes to complete the mission, averaged only over those missions which are completed successfully. In other words, the question arises: If we condition that A is reached before B, what is then the expected time to reach A?

Although this question seems almost as basic as the two previous ones, we have not been able to find it answered explicitly in the literature. In this note we show that - still, under suitable technical assumption - this conditional expectation of the first passage time can be computed as

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} = \frac{\kappa(x)}{\psi(x)}$$

where ψ is the probability that A is reached before B, as above, and where κ is the unique solution to the equation

$$L\kappa = -\psi, \quad \kappa|_{\partial\Omega} = 0$$

This is our main result which is stated precisely and proved in section A.1 below. In section A.2 we state a rather straightforward generalization where a reward is released upon first passage; making this reward equal to the time of first passage recovers the result of section A.1. In section A.3 we show how one may obtain upper bounds if given solutions to the corresponding partial differential *inequalities*. This is especially useful in situations where the partial differential *equations* have no (classical) solutions which will be the case in many applications.

Notation

Our notation is fairly standard and follows [83]. In particular, the diffusions we consider in this note are Itô diffusions evolving in Euclidean space $\mathbb{X} = \mathbb{R}^n$ according to the stochastic differential equation

$$dx_t = f(x_t)\ dt + g(x_t)\ dB_t \tag{A.1}$$

which we interpret in the Itô sense. Of course, we assume an underlying filtered probability space which we however do not refer to explicitly.

We define the (backward) differential operator L associated with the diffusion x in the usual way: If $V : \mathbb{X} \to \mathbb{R}$ is C^2, then

$$LV(x) = V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g)$$

where the right hand side is evaluated at x.

If $D \subset \mathbb{X}$ is Borel then we use τ_D to denote the stopping time $\inf\{t > 0 : x_t \in D\}$. P^x is the probability law of x_t starting at $x_0 = x$ and E^x denotes expectation w.r.t. P^x.

For a set A, $\bar{A}$ denotes the closure of A.

If A is an event such that $P^x A > 0$ and y is a stochastic variable for which $E^x|y| < \infty$, then $E^x\{y \mid A\}$ denotes the conditional expectation $E^x\{y \mid \mathcal{A}\}$ evaluated at some $\omega \in A$; here $\mathcal{A}$ denotes the σ-algebra generated by A.

A.1 The main result

We make the following assumptions on the geometry and the dynamics:

Assumption 75:

- i The initial condition x of the stochastic differential equation is in a domain $\Omega \subset \mathbb{X}$ which is open and bounded and has a smooth boundary $\partial\Omega$.
- ii The drift coefficient f and diffusion coefficient g are Lipschitz continuous on the closure $\bar{\Omega}$ of the domain.

iii The diffusion g satisfies the non-degeneracy condition that $gg' > 0$ on $\bar{\Omega}$.

iv The boundary $\partial\Omega$ is divided into two disjoint components A and B which have no common limit points, i.e. $A \cup B = \partial\Omega$ and $\bar{A} \cap \bar{B} = \emptyset$.

□

These assumptions are standard and natural from a classical point of view: The Lipschitz continuity and the boundedness of Ω assure that there exists a unique solution of the stochastic differential equation at least up to the first time the boundary $\partial\Omega$ is reached, see [83]. The non-degeneracy condition on g ensures that the first passage time $\tau_{\partial\Omega}$ is finite w.p. 1 and has finite expectation. It also implies that L is uniformly elliptic which gives us existence and uniqueness of solutions in the classical sense to the partial differential equations we consider. The condition that $\bar{A}$ and $\bar{B}$ are disjoint implies that the probability $\psi(x) = P^x\{\tau_A = \tau_{\partial\Omega}\}$ is Lipschitz continuous on $\partial\Omega$ and hence C^2 on Ω.

Later, we relax some of the assumptions somewhat.

Our main result is the following:

Theorem 76: For the diffusion (A.1) under assumption 75, we have the following formula for the conditional expectation of the first passage time

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} = \frac{\kappa(x)}{\psi(x)}$$

for any point $x \in \Omega$ such that $\psi(x) > 0$. Here $\psi(x)$ equals $P^x\{\tau_A = \tau_{\partial\Omega}\}$ and is the unique solution to the equation

$$L\psi = 0, \quad \psi|_A = 1, \quad \psi|_B = 0 \tag{A.2}$$

while $\kappa : \mathbb{X} \to \mathbb{R}$ is the unique solution to the equation

$$L\kappa = -\psi, \quad \kappa|_{\partial\Omega} = 0 \quad . \tag{A.3}$$

□

Proof: It is well known [107, chpt. 3] that the assumptions imply that $\psi(x) = P^x\{\tau_A = \tau_{\partial\Omega}\}$ is C^2 and the unique solution to (A.2); a compact exposition of the necessary results can be found in [40, sec. 3.5]. This in

turn implies that κ is well defined as the unique solution to (A.3). Furthermore, $\tau_{\partial\Omega}$ is finite w.p. 1 and has finite expectation which implies that the conditional expectation is well defined [83, p. 239].

For $s \in \mathbb{R}$, define the process $s_t = s + t$. Let $y_t = (x_t, s_t)$; then y_t is the unique (up to $\tau_{\partial\Omega}$) solution to the stochastic differential equation

$$dy_t = \begin{pmatrix} f(y_t)\ dt + g(y_t)\ dB_t \\ 1\ dt \end{pmatrix} \quad .$$

We stop the process y_t when it hits $\partial\Omega \times \mathbb{R}$ (i.e, at $t = \tau_{\partial\Omega}$) and define the reward function for $y = (x, s)$

$$\lambda(y) = s \cdot \chi_A(x) = \begin{cases} s & \text{if } x \in A \\ 0 & \text{else.} \end{cases}$$

Define the expected reward

$$\nu(y) = E^y \lambda\left(y(\tau_{\partial\Omega})\right)$$

and let $y = (x, s)$ with $x \in \Omega$, then

$$\begin{aligned} \nu(y) &= E^y\left\{\lambda\left(y(\tau_{\partial\Omega})\right) \mid \tau_A = \tau_{\partial\Omega}\right\} \cdot P^y\{\tau_A = \tau_{\partial\Omega}\} \\ &+ E^y\left\{\lambda\left(y(\tau_{\partial\Omega})\right) \mid \tau_B = \tau_{\partial\Omega}\right\} \cdot P^y\{\tau_B = \tau_{\partial\Omega}\} \\ &= E^x\{s + \tau_A \mid \tau_A = \tau_{\partial\Omega}\} \cdot P^x\{\tau_A = \tau_{\partial\Omega}\} \\ &= (s + E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\}) \cdot \psi(x) \end{aligned}$$

Define the (backward) differential operator M associated with the diffusion y in the usual way: If $W : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ is $C^{2,1}$ then

$$MW(y) = W_x f + W_t + \frac{1}{2}\mathrm{tr}(g' W_{xx} g)$$

where the right hand side is evaluated at $y = (x, s)$. Then $M\nu = 0$ on $\Omega \times \mathbb{R}$. Furthermore, $\nu(x, s) = s$ for $x \in A$ and $\nu(x, s) = 0$ for $x \in B$.

We claim that $\nu(x, s) = \kappa(x) + s \cdot \psi(x)$. To see this notice that $\nu_t(x, s) = \psi(x)$. Together with $M\nu = 0$ and the boundary conditions this implies that $\nu(x, 0) = \kappa(x)$ on $\bar{\Omega}$ from which the conclusion follows.

Combining the above expressions yields

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} \cdot \psi(x) = \nu(x, s) - s \cdot \psi(x) = \kappa(x)$$

which completes the proof. ■

Example 77: [Brownian motion] Consider the case of scalar Brownian motion, i.e. $\mathbb{X} = \mathbb{R}$ and $f = 0$, $g = 1$. Let Ω be the open interval $(0, 1)$ and let $A = \{1\}$, $B = \{0\}$. Then $\psi(x) = x$ and $\kappa(x) = \frac{1}{3}x(1 - x^2)$, i.e.

$$E^x\{\tau_A \mid \tau_{\partial\Omega} = \tau_A\} = \frac{1}{3}(1 - x^2) \text{ for } 0 < x \leq 1 \quad . \tag{A.4}$$

For comparison we have the unconditional expectation

$$E^x \tau_{\partial\Omega} = x - x^2 \text{ for } 0 \leq x \leq 1 \quad .$$

Notice that the conditional expectation and the unconditional expectation coincide for $x = 1/2$ as symmetry predicts.

Figure A.1 shows numerical results which are obtained in the following way: For each initial condition in $\{0.05,\ 0.10,\ \ldots\ , 0.95\}$, we perform a number of simulations until we obtain 100 simulations which exit to the right. Simulations are done with a sample time of $\Delta t = 0.0001$. For these 100 simulations we compute and plot the average first exit time (marked with $\times$ in the figure). The sample means are slightly larger than the conditional expectation as computed by the expression (A.4) (the solid line in the figure). The difference decreases with the sample time Δt (although the plot shows results for only one sample time). This is to be expected: When we only observe the diffusion at discrete points of time we only get an upper bound on the first exit time, and sample paths starting near $x = 0$ are prone to misclassification. □

A.2 A generalization

A way to generalize the result from the previous section is to see that the first passage time is a functional on the set of trajectories and then consider more general functionals. The functionals we consider in this section consist of two components: A cumulative term, i.e. an integral along the trajectory, and a terminal term depending on where the the trajectory hits the boundary. More specifically, we obtain a formula for

$$E^x\left\{k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \;\middle|\; \tau_A = \tau_{\partial\Omega}\right\} \quad .$$

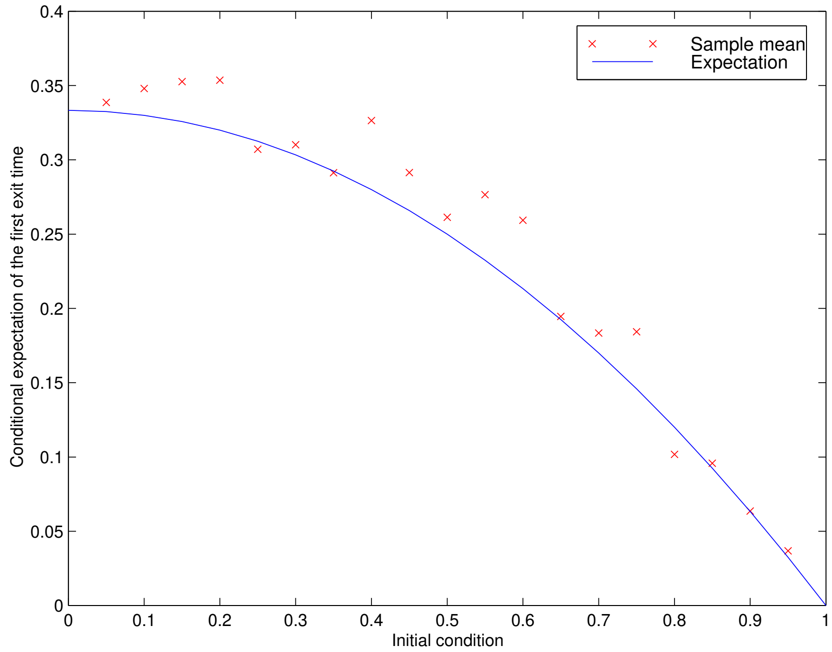

Figure A.1: Numerical results obtained for scalar Brownian motion

Assumption 78:

i: The function $l : \bar{\Omega} \to \mathbb{R}$ is Lipschitz continuous.

ii: The function $k : \partial\Omega \to \mathbb{R}$ is Lipschitz continuous.

□

Theorem 79: For the diffusion (A.1) with the reward functions k, l under assumptions 75, 78, we have the following formula

$$E^x \left\{ k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \ \middle|\ \tau_A = \tau_{\partial\Omega} \right\} = \frac{\kappa(x)}{\psi(x)}$$

for any point $x \in \bar{\Omega}$ such that $\psi(x) > 0$. Here $\psi(x)$ is as before and κ is the unique solution to the partial differential equation

$$L\kappa = -l \cdot \psi, \quad \kappa|_A = k, \quad \kappa|_B = 0 \quad .$$

□

Proof: As in the proof for the previous theorem existence and uniqueness of a solution κ to the partial differential equation is guaranteed; notice that the boundary condition $k \cdot \chi_A$ is Lipschitz continuous since k is and since $\bar{A}$ and $\bar{B}$ are disjoint. Also, the conditional expectation is well defined.

Define $y_t = (x_t, z_t)$ where z_t solves the stochastic differential equation

$$dz_t = l(x_t)\ dt \quad .$$

Existence and uniqueness of a solution to this equation is guaranteed since l is Lipschitz continuous. Let $z_0 = z$ be the corresponding initial condition and define the reward

$$\lambda(x, z) = (k(x) + z)\chi_A(x)$$

and the expected reward

$$\nu(x, z) = E^{x,z}\lambda(x(\tau_{\partial\Omega}), z(\tau_{\partial\Omega}))$$

Again, we define the backward differential operator M associated with $y = (x, z)$ in the usual way: If $W : \mathbb{X} \times \mathbb{Z} \to \mathbb{R}$ is $C^{2,1}$, then

$$MW(y) = W_x f + W_z l + \frac{1}{2}\mathrm{tr}(g' W_{xx} g)$$

where the right hand side is evaluated at $y = (x, z)$. Then $M\nu = 0$ on Ω and $\nu = \lambda$ on $\partial\Omega$. Following the proof of theorem 76, we see that $\nu_z = \psi$ on Ω and hence that $\nu(x, z) = \kappa(x) + z \cdot \psi(x)$. Finally we notice that

$$\nu(x, z) = E^{x,z}\{\lambda(x(\tau_{\partial\Omega}), z(\tau_{\partial\Omega})) \mid \tau_{\partial\Omega} = \tau_A\} \cdot \psi(x)$$

which completes the proof. ■

A.3 An upper bound under weak assumptions

A weakness of the previous results is that the assumptions are rather restrictive. In particular, we would like to allow for non-smooth boundaries, degenerate diffusion coefficients and situations where $\bar{A}$ and $\bar{B}$ are not disjoint (although A and B are). This means that we must obtain the desired results without having guaranteed existence and uniqueness of solutions to the involved partial differential equations. For instance, if $\bar{A}$ and $\bar{B}$ are not disjoint then ψ cannot be continuous on $\bar{\Omega}$. This motivates us to establish results which guarantees *bounds* through partial differential inequalities.

In this section we use $\bar{x}_t$ to denote the process x_t stopped at $\partial\Omega$, i.e. $\bar{x}(t) = x(t \wedge \tau_{\partial\Omega})$.

Assumption 80:

i : The domain Ω is open and bounded.

ii : The drift coefficient f and the diffusion coefficient g are Lipschitz continuous on $\bar{\Omega}$.

iii : The boundary $\partial\Omega$ is reached in finite time, almost surely, and furthermore $E^x\tau_{\partial\Omega} < \infty$ for all $x \in \Omega$.

iv : The boundary $\partial\Omega$ of the domain is divided into two disjoint Borel sets A and B, i.e. $A \cup B = \partial\Omega$ and $A \cap B = \emptyset$.

v : $k : \partial\Omega \to \mathbb{R}$ and $l : \bar{\Omega} \to \mathbb{R}$ are Lipschitz continuous and non-negative.

□

The assumption that $E^x\tau_{\partial\Omega} < \infty$ is not always immediate; in these situations one can use the sufficient condition that there exists a C^2 function $\phi : \bar\Omega \to \mathbb{R}$ such that $L\phi < 0$ on $\bar\Omega$.

We start off with an elementary lemma; many similar statements can be found in the literature.

Lemma 81: Let assumption 80 hold and let $\bar\psi : \bar\Omega \to \mathbb{R}$ be C^2 and satisfy

$$L\bar\psi \leq 0, \quad \bar\psi \geq 0, \quad \bar\psi|_A \geq 1 \quad .$$

Then the bound

$$\psi(x) \leq \bar\psi(x)$$

holds. Conversely, let $\underline{\psi} : \bar\Omega \to \mathbb{R}$ be C^2 and satisfy

$$L\underline{\psi} \geq 0, \quad \underline{\psi} \leq 1, \quad \underline{\psi}|_B \leq 0 \quad .$$

Then the bound

$$\psi(x) \geq \underline{\psi}(x)$$

holds. □

Proof: The assumptions imply that $\bar\psi(\bar x_t)$ is an almost surely continuous non-negative supermartingale. We then have the inequalities

$$P^x\{\tau_A = \tau_{\partial\Omega}\} \leq P^x\{\sup_{t\geq 0} \bar\psi(\bar x_t) \geq 1\} \leq \bar\psi(x)$$

using Doob's martingale inequality, see e.g. [83, p. 28]. The converse statement follows similarly after noting that $1 - \underline{\psi}(\bar x_t)$ is a non-negative supermartingale; here we must use that the process exits Ω in finite time, almost surely. ■

Theorem 82: Let assumptions 80 hold and let $\bar\kappa$ be a non-negative C^2 function $\bar\Omega \to \mathbb{R}$ which satisfies

$$L\bar\kappa \leq -l \cdot \bar\psi, \quad \bar\kappa|_{\partial\Omega} \geq k \cdot \chi_A$$

where $\bar\psi$ is as in lemma 81. Let $\underline{\psi}$ satisfy $\underline{\psi} \leq \psi$ on $\bar\Omega$. Then the bound

$$E^x\left\{k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \;\middle|\; \tau_A = \tau_{\partial\Omega}\right\} \leq \frac{\bar\kappa(x)}{\underline{\psi}(x)}$$

holds at any point $x \in \bar\Omega$ for which $\underline{\psi} > 0$. □

Proof: As in the previous existence of the conditional expectation is guaranteed. Let z_t be the unique solution to the stochastic differential equation

$$dz_t = l(x_t)\ dt$$

with initial condition $z_0 = z \in \mathbb{R}$. We let $\bar z(t)$ denote the stopped process $\bar z_t = z(t \wedge \tau_{\partial\Omega})$.

Define

$$\bar\nu(x,z) = \bar\kappa(x) + z \cdot \bar\psi(x)$$

Then we have

$$M\bar\nu = L\bar\kappa + z \cdot L\bar\psi + l \cdot \bar\psi \leq 0$$

for any x, z with $x \in \Omega$ and $z \geq 0$. Notice that if the initial condition z is non-negative, then so is $\bar z_t$ for $t \geq 0$ since $l \geq 0$. This implies that $\bar\nu(\bar x_t, \bar z_t)$ is a non-negative supermartingale with continuous sample paths, almost surely, which in turn implies that the inequality

$$E^{x,z}\{\bar\nu(\bar x(\tau_{\partial\Omega}), \bar z(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \cdot P^x\{\tau_A = \tau_{\partial\Omega}\} \leq \bar\nu(x,z)$$

holds. Manipulating the left hand side we obtain

$$\begin{aligned}
& E^{x,z}\{\bar\nu(\bar x(\tau_{\partial\Omega}), \bar z(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
\geq\ & E^{x,z}\{z(\tau_{\partial\Omega}) + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
=\ & z + E^{x,0}\{z(\tau_{\partial\Omega}) + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
=\ & z + E^x\{\int_0^{\tau_{\partial\Omega}} l(x_s)\ ds + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \quad .
\end{aligned}$$

We have thus shown that

$$E^x\left\{\int_0^{\tau_{\partial\Omega}} l(x_s)\ ds + k(x(\tau_{\partial\Omega})) \;\middle|\; \tau_A = \tau_{\partial\Omega}\right\} \cdot \psi(x) \leq \bar\nu(x,0) = \bar\kappa(x)$$

holds. The result follows. ■

A.4 Numerical issues

Under the assumptions 75 and 78 there exists smooth solutions to the involved partial differential equations and standard methods for their solution can be employed.

Example 83: [Two-dimensional Brownian motion] Consider the case $n = 2$, $dx_t = dB_t$ and let the domain Ω be

$$\Omega = \{x \in \mathbb{R}^2 \mid \|x\|_\infty < 1 \ \wedge \ \|x\|_\infty > \frac{1}{4}\}$$

Let $A \subset \partial\Omega$ be the outer boundary, i.e. $A = \{x \mid \|x\|_\infty = 1\}$. The operator L is then $\Delta/2$. Using a quadratic grid with a step length of 0.05, we have discretized the partial differential equations using a finite difference method. The solutions are seen in figures A.2 through A.4. □

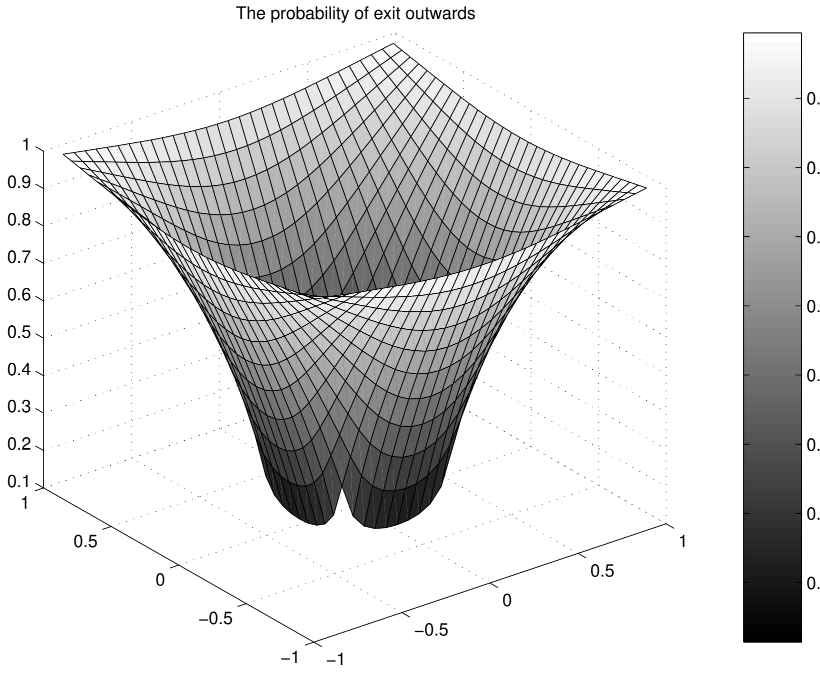

Figure A.2: 2-D Brownian motion: Probability of exit outwards

Under the weaker assumptions 80 one has to consider carefully if the partial differential equations have solutions in the classical sense. One option is to approximate the problem with one which satisfies the assumptions 75. For instance when $\bar{A} \cap \bar{B} \neq \emptyset$ one may choose to approximate χ_A with a function

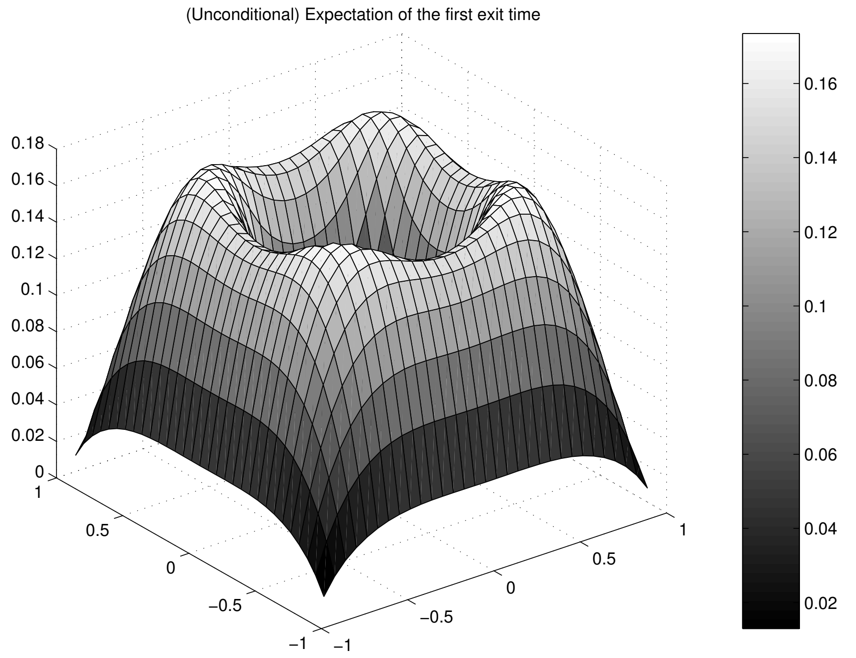

Figure A.3: 2-D Brownian motion: Unconditional expectation of the first exit time

which is Lipschitz continuous on $\partial\Omega$. The weak maximum principle, see e.g. [91, p. 106], is useful for establishing relations between approximated solutions obtained in this fashion.

An alternative is to search for solutions to the partial differential inequalities of section A.3 in some finite dimensional subspace, for instance spun by trigonometric functions or polynomials. If one only requires that the inequalities are satisfied at some finite set of points in $\bar{\Omega}$, then the problem of finding the best bounding functions $\bar{\psi}$, $\underline{\psi}$ and $\bar{\kappa}$ becomes one of linear programming.

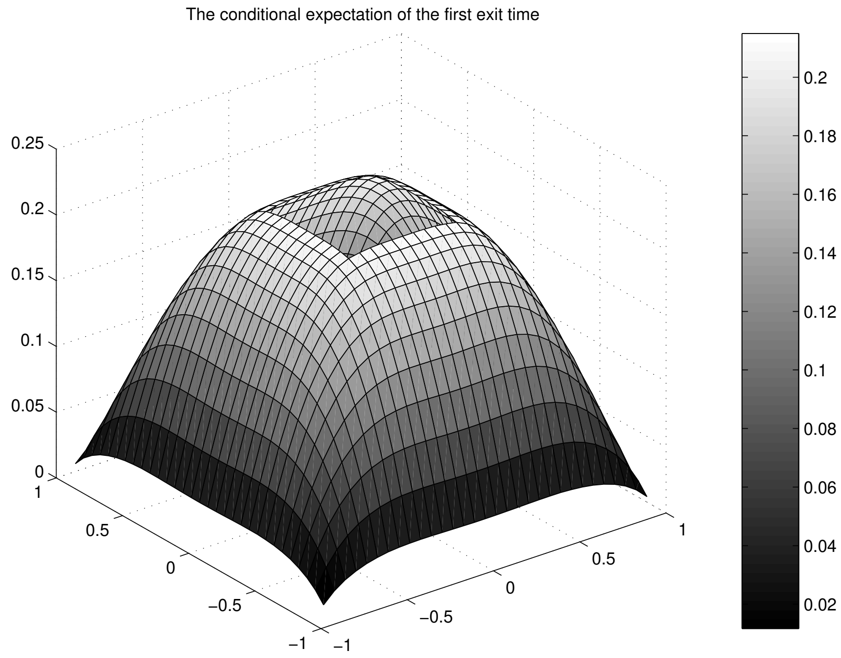

Figure A.4: 2-D Brownian motion: Conditional expectation of the first exit time

A.5 Summary

For solutions to a stochastic differential equation starting is some bounded domain, we have derived a formula for the conditional expectation of the time of first exit from the domain. The conditioning is with respect to the event that a given part of the boundary is reached first. The formula requires the solution of two elliptic partial differential equations. We have also provided a generalization to other functionals than the first time of exit, and we have established bounds which are expressed in terms of partial differential inequalities.

We have concentrated on classical (i.e, C^2) solutions to the involved partial differential equations and inequalities as well as classical conditions for existence and uniqueness of solutions to the equations. Similar results can

be obtained under weaker hypothesis if one employs the notion of viscosity solutions and uses the results of [84]. This is a topic of current research; the results will appear in [112].

Appendix B

Various technicalities

This appendix contains various proofs and calculations which are not essential for the understanding of the results in this thesis.

B.1 Proof of theorem 25 on page 60

Due to the condition (3.11) we know that there exists parameters $\lambda_j(t) \geq 0$, $j = 1, \ldots, m$, such that

$$[A(t), B(t), C(t), D(t)] = \sum_{j=1}^{m} \lambda_j(t)[A_j, B_j, C_j, D_j], \qquad \sum_{j=1}^{m} \lambda_j(t) = 1$$

We omit the time argument after signals and use the notation

$$z_j = C_j x + D_j w$$

Our candidate storage function for Σ is $x'Px$. We then get

$$\begin{aligned}
\frac{d}{dt} x'Px &= (x' \quad w') \begin{bmatrix} PA(t) + A'(t)P & PB(t) \\ B'(t)P & 0 \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\
&= (x' \quad w') \left(\sum_{j=1}^{m} \lambda_j \begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} \right) \begin{pmatrix} x \\ w \end{pmatrix}
\end{aligned}$$

$$\leq \sum_{j=1}^{m} \lambda_j (x' \quad w') \left(\sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix} - \epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} [C_j \ D_j] \right) \begin{pmatrix} x \\ w \end{pmatrix}$$

$$= \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{p} d_i (w' \quad z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} - \epsilon z_j' z_j \right)$$

$$= \sum_{i=1}^{p} d_i \left(\sum_{j=1}^{m} \lambda_j (w' \quad z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} \right) - \epsilon \sum_{j=1}^{m} \lambda_j z_j' z_j$$

$$\leq \sum_{i=1}^{p} d_i \left((w' \quad z') Q_i \begin{pmatrix} w \\ z \end{pmatrix} \right) - \epsilon z' z$$

We have thus show that the time-invariant function $x'Px$ is a strong storage function for the time-varying system Σ w.r.t. the supply rate $\sum_i d_i s_i$ and hence we may conclude robust stability of the interconnection (Σ, Δ).

B.2 The filter ODE for the conditional state estimate

In this appendix we derive the filter ODEs (4.13) and (4.14) for the conditional worst case state estimate $\xi(i,t)$ and the associated loss $S(\xi(i,t),i,t)$. The derivation follows the general procedure of [120].

The loss function $S(x,i,t)$ is quadratic in x. This means, that the characterization of the worst-case conditional state estimate $\xi(i,t)$ is

$$\frac{\partial}{\partial x} S(\xi(i), i, t) = 0 \tag{B.1}$$

and

$$\frac{\partial^2}{\partial x^2} S(\xi(i), i, t) > 0 \quad .$$

At this point we omit the x and t arguments and adopt the simplified notation S_x for $\frac{\partial}{\partial x} S(x,i,t)$ and so forth.

The cost-to-go and cost-to-come satisfy the PDEs [9, 120]:

$$P_x A_i x + \frac{1}{2} P_x \left(\frac{1}{\gamma^2} G_i G_i' - B_i B_i' \right) P_x' + \frac{1}{2} x' H_i' H_i x = 0 \tag{B.2}$$

and

$$R_t + R_x (A_i x + B_i u) + \frac{1}{2\gamma^2} R_x G_i G_i' R_x' \tag{B.3}$$

$$+ \frac{1}{2} x' H_i' H_i x - \frac{1}{2}\gamma^2 \|y - C_i x\|^2 + \frac{1}{2}\|u\|^2 = 0 \quad .$$

The PDE (B.2) reduces to the control algebraic Riccati equation (4.5) after guessing P to be quadratic in x for fixed i. Likewise, the PDE (B.3) is related to the filter algebraic Riccati quation (4.12): Guess R to be quadratic in x for fixed i, assume stationarity in the sense $R_{xxt} = 0$ and consider only second order terms in (B.3). For each i, define $Q_i := R_{xx}(x,i)$, then Q_i must satisfy the ARE (4.12).

Using $S = R - P$ we get by subtracting (B.2) from (B.3) and rearring terms

$$S_t + \frac{1}{2}\|u + B_i' P_x'\|^2 - \frac{1}{2}\gamma^2 \|y - C_i x\|^2 + S_x (A_i x + B_i u) \tag{B.4}$$

$$+ \frac{1}{2\gamma^2} (S_x + 2P_x) G_i G_i' S_x' = 0 \quad .$$

This must in particular hold for $x = \xi(i,t)$. Using the stationarity condition (B.1) we then find the ODE for the conditional loss (4.14)

$$\frac{d}{dt} S(\xi(i,t), i, t) = \frac{1}{2}\gamma^2 \|y - C_i \xi(i,t)\|^2 - \frac{1}{2}\|u + B_i' P_x'\|^2 \quad .$$

If the parameter estimate $\hat{\theta}(t) = \arg\min_i S_i(t)$ is well defined, then we may use the the certainty equivalence control $u(t) = -B_{\hat{\theta}(t)}' P_x'(\xi(\hat{\theta}(t),t), \hat{\theta}(t))$ to obtain that the *unconditional* worst-case loss (for $i = \hat{\theta}(t)$) satisfies

$$\frac{d}{dt} S(\xi(\hat{\theta}(t),t), \hat{\theta}(t), t) = \frac{1}{2}\gamma^2 \|y - C_{\hat{\theta}(t)} \xi(\hat{\theta}(t), t)\|^2$$

and hence is non-decreasing.

To obtain the observer equation (4.13), we again follow [120] and differentiate the stationarity condition with respect to t to get

$$\frac{d}{dt} S_x(\xi(i,t), t) = 0 \;\Leftrightarrow\; \dot{\xi}(i,t) = -\left. S_{xx}^{-1} S_{tx}' \right|_{x=\xi(i,t)} \quad .$$

The expression for S_{tx} is found by viewing (B.4) as a relation between x, t, u and y and differentiating with respect to x. Using the stationarity condition to eliminate the terms including S_x we get:

$$S_{tx} + (u + B_i' P_x')' (B_i' X_i) + \gamma^2 (y - C_i \xi)' C_i$$

$$+\left(\xi' A_i' + u' B_i' \frac{1}{\gamma^2} P_x G_i G_i\right)' S_{xx} = 0$$

which must hold for all t, u, y at $x = \xi(i,t)$. Combining, we obtain

$$\dot{\xi}(i,t) = A_i\xi + B_i u + \frac{1}{\gamma^2} G_i G_i' P_x' + S_{xx}^{-1}\gamma^2 C_i'(y - C_i\xi) + S_{xx}^{-1} X_i B_i (u + B_i' P_x') \quad .$$

This may also be written as (4.13):

$$\dot{\xi}(i,t) = A_i\xi - B_i B_i' P_x' + \frac{1}{\gamma^2} G_i G_i' P_x' + S_{xx}^{-1}\gamma^2 C_i'(y - C_i\xi) + S_{xx}^{-1} R_{xx} B_i (u + B_i' P_x')$$

Appendix C

Frequently used symbols and acronyms

Miscellaneous

A'	Complex conjugate transpose of matrix A
$\boldsymbol{\Delta}$	A set of dynamic state space systems (perturbations)
Σ	Nominal system
Δ	Perturbation
(Σ, Δ)	Perturbed system; interconnection of Σ and Δ
τ_D	Stopping time; first exit from domain D

Functions and operators

$\operatorname{Arg\,min}_x f(x)$	The set $\{x \mid f(x) = \inf_\xi f(\xi)\}$ where $f(x) \in \mathbb{R}$
$\arg\min_x f(x)$	The unique element of $\operatorname{Arg\,min}_x f(x)$
$o(\delta)$	A function for which $\|\|o(\delta)\|\|/\|\|\delta\|\| \to 0$ as $\|\|\delta\|\| \to 0$
V_x	Gradient of C^1 function V, $\partial V/\partial x$
V_{xx}	Hessian of C^2 function V
$V^{-1}(A)$	Preimage of A under V, i.e. $\{x \mid V(x) \in A\}$
$LV(x)$	Backwards operator of an autonomous diffusion
$L^u V(x)$	Backwards operator of a controlled diffusion

Sets and spaces

$[a,b]$, (a,b), $[a,b)$, $(a,b]$	Closed, open, and half-open real intervals
$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	Real, natural, integer numbers
$\mathbb{R}_+, \mathbb{R}_-$	Positive, negative real numbers
$\bar{A}$, $\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_-$	Closure of sets
A^o, ∂A	Interior, boundary of set A
$\mathbf{Co}(A)$	Convex hull of a set A in a linear space
$\mathbb{X}$	State space, typically $\mathbb{R}^n$
$T\mathbb{X}, T^*\mathbb{X}$	Tangent and cotangent bundle of $\mathbb{X}$
$\mathcal{L}_2(\mathbb{X}, \mathbb{Y})$	Lebesque space of square integrable functions from $\mathbb{X}$ to $\mathbb{Y}$
$\mathcal{H}_\infty$	Hardy space of complex functions, analytical in the closed right half plane

Acronyms

CE	Certainty equivalence
FSN	Finite signal-to-noise ratio
HJ	Hamilton-Jacobi
HJB	Hamilton-Jacobi-Bellman
HJI	Hamilton-Jacobi-Isaacs
LMI	Linear matrix inequality
ODE	Ordinary differential equation
PDE	Partial differential equation
PDI	Partial differential inequality
SDE	Stochastic differential equation

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