Bloom Filters

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Algorithmic Techniques for Modern Data Models
DTU

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Overview for today

- Independent random variables

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- Hash functions for Bloom filters

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- Problem definition

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- Comparison to lower bound

Independent random variables

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  - For instance, for any $x, y \in U$:

$$\mathrm{P}[h_1(x) = 2, h_2(y) = 4] = \mathrm{P}[h_1(x) = 2] \cdot \mathrm{P}[h_2(y) = 4]$$

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  - Answer a query of the form "Is $x \in X$?" for any query element $x \in U$

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- We assume the hash functions have the properties stated earlier (uniformity, independence)

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Index:	1														$m = 15$

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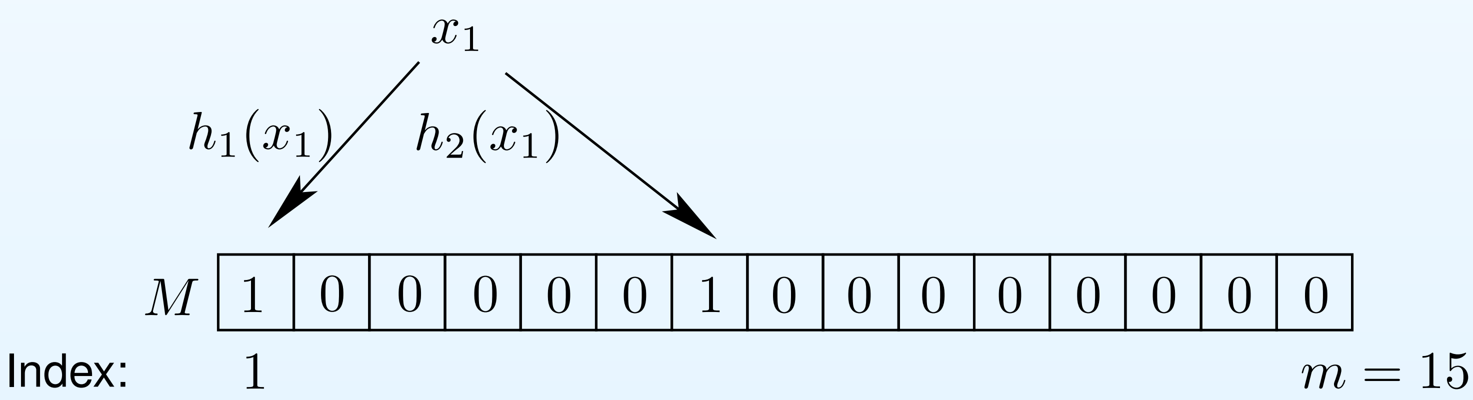

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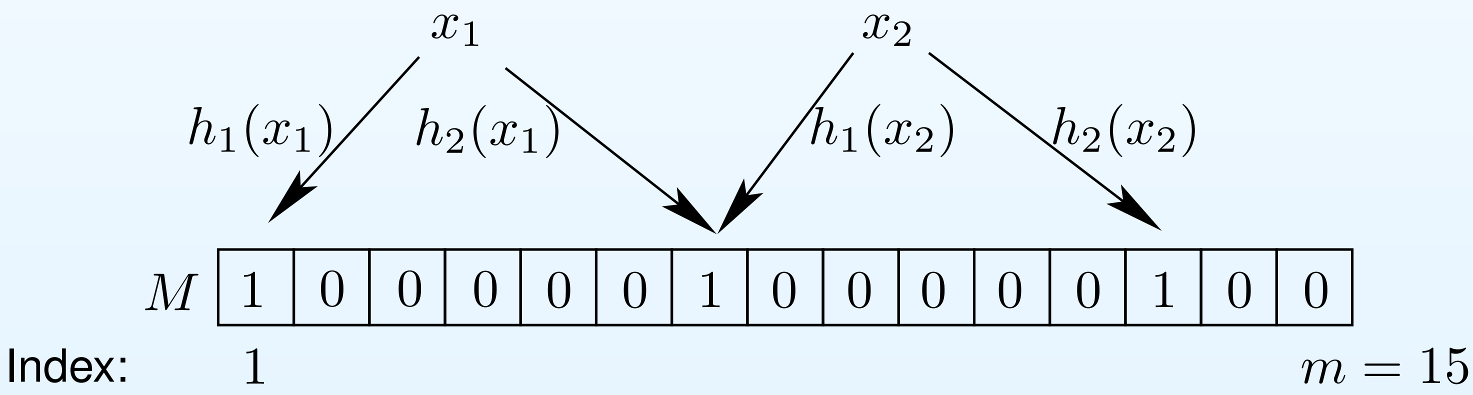

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- We therefore focus on analyzing queries

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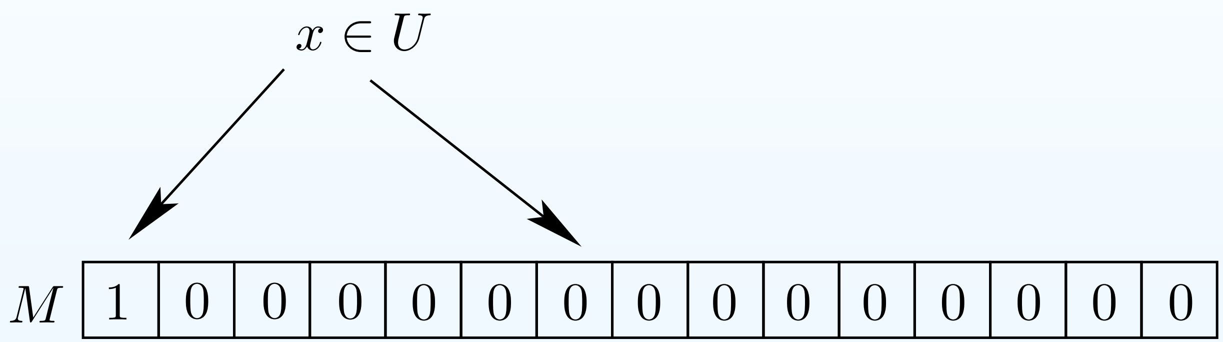

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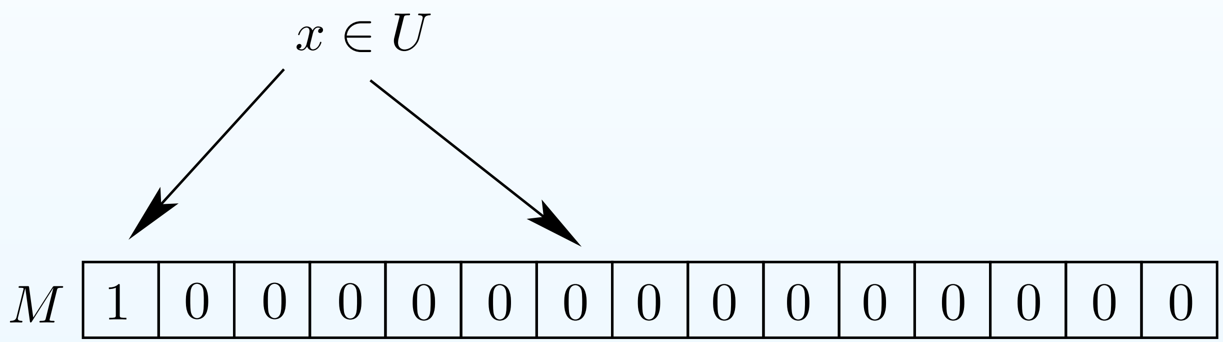

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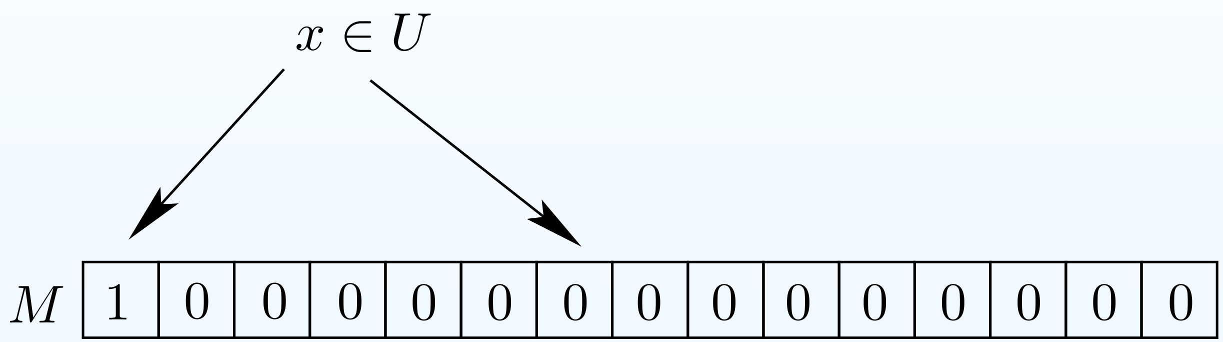

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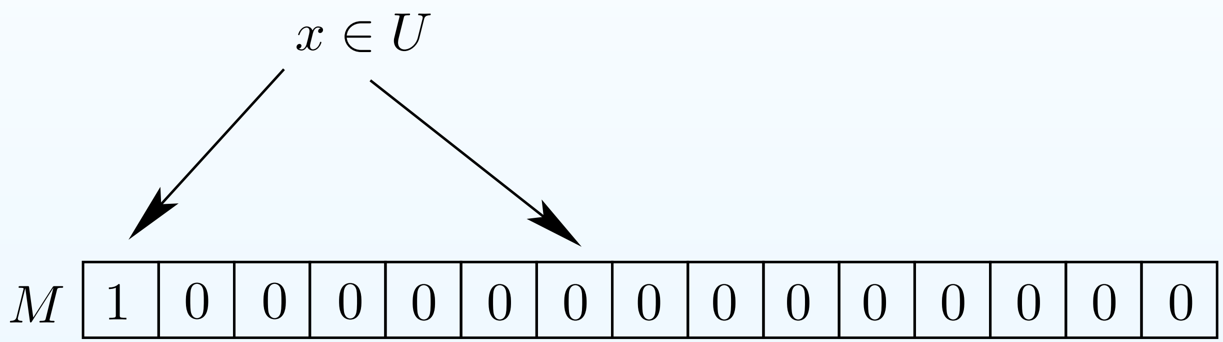

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- Conclusion: the Bloom filter is always correct when it answers “No”

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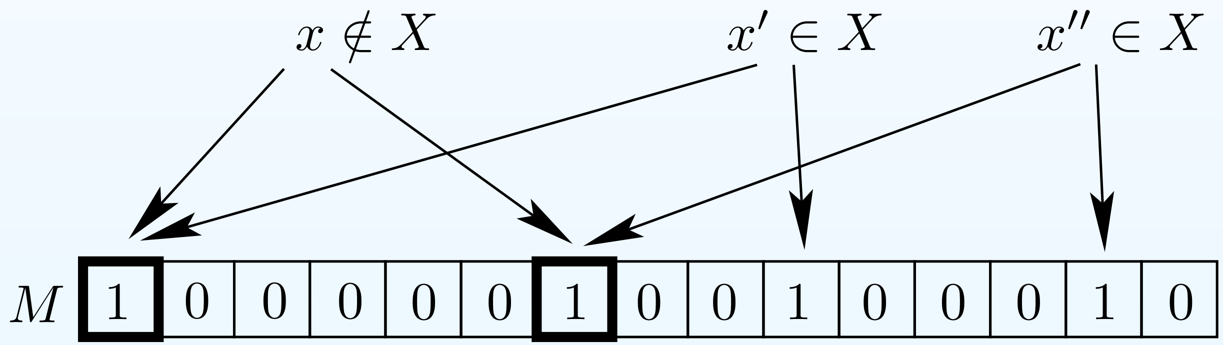

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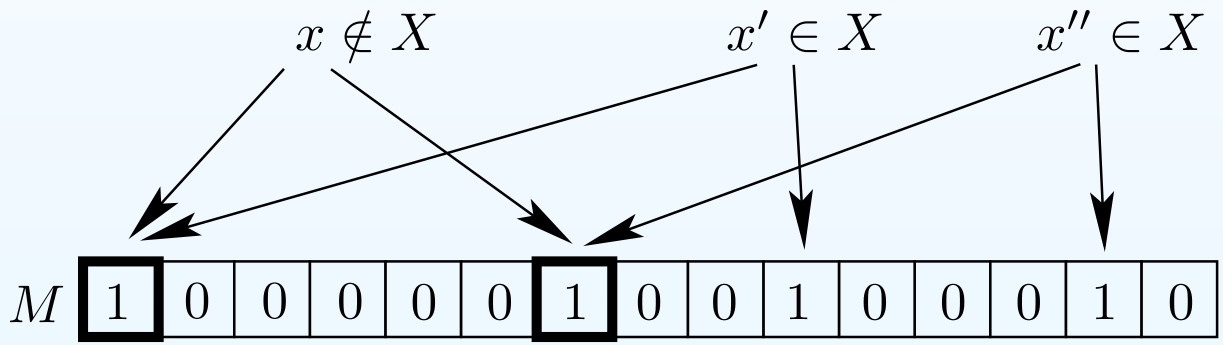

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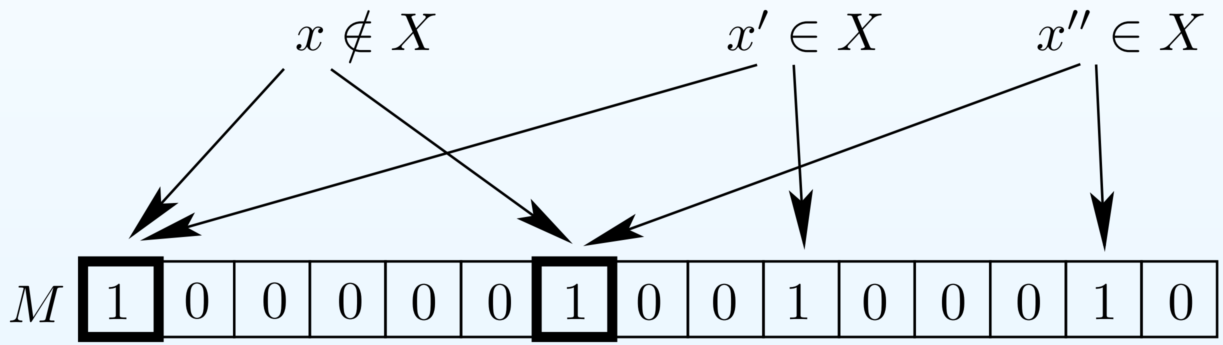

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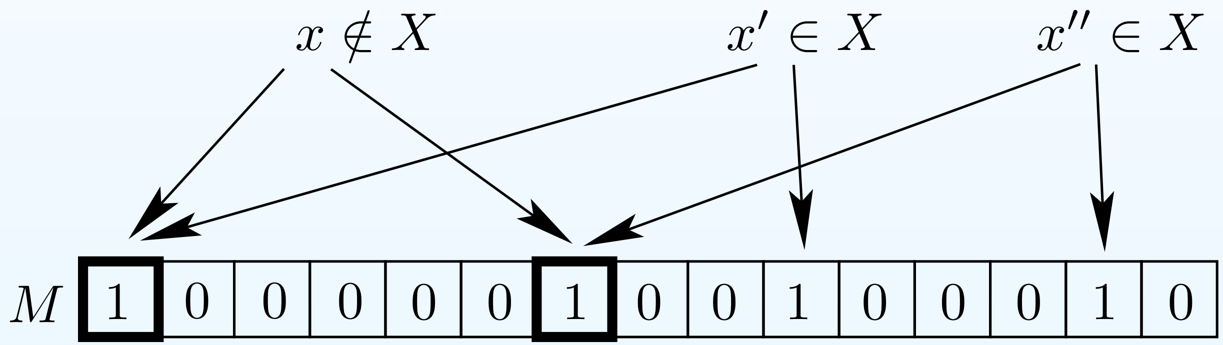

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$$\rho = \frac{10}{15} = \frac{2}{3}$$

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- Using independence of $h_1(x), \ldots, h_k(x)$, the probability that the Bloom filter incorrectly answers "Yes" for x is thus

$$\mathrm{P}[M[h_1(x)] = 1, \ldots, M[h_k(x)] = 1] = (1 - \rho)^k$$

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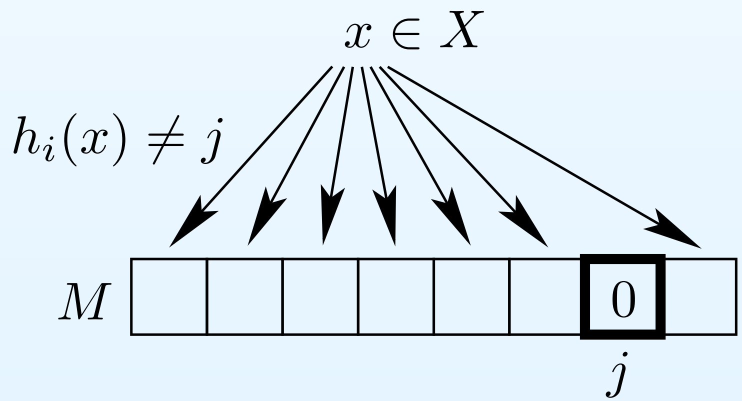

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- Thus, the probability of a false positive is

$$(1-\rho)^k \approx (1-p')^k = \left(1 - \left(1 - \frac{1}{m}\right)^{kn}\right)^k$$

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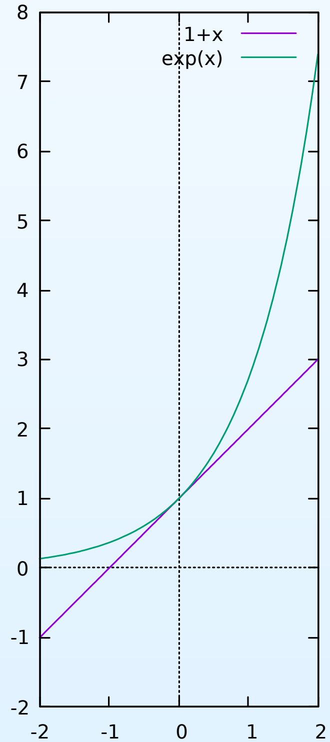

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- Setting $x=-1/m$, the approximate probability then becomes:

$$\left(1-\left(e^{-\frac{1}{m}}\right)^{kn}\right)^{k}=\left(1-e^{-\frac{kn}{m}}\right)^{k}=\exp(k\ln(1-e^{-\frac{kn}{m}}))$$

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- We simplify this with the approximation $1 + x \approx e^x$ for x close to 0
- Setting $x = -1/m$, the approximate probability then becomes:

$$\left(1 - \left(e^{-\frac{1}{m}}\right)^{kn}\right)^k = \left(1 - e^{-\frac{kn}{m}}\right)^k = \exp(k \ln(1 - e^{-\frac{kn}{m}}))$$

- We want to pick k to minimize this expression

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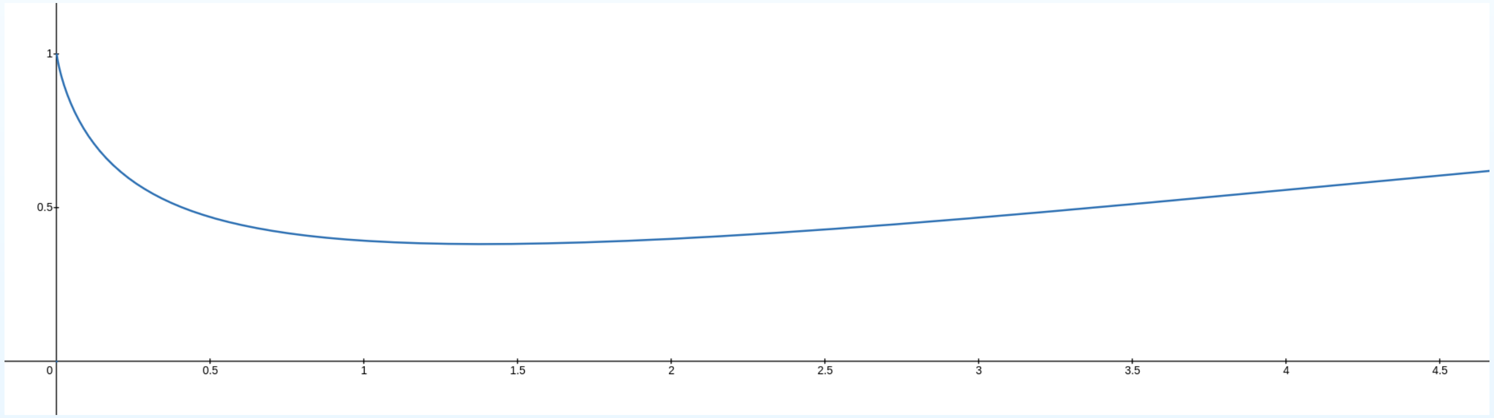
- Example with $m = 2n$ and minimum at $k_{\min} = 2 \ln 2 \approx 1.386$:



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- Equivalently, this is

$$2^{-\frac{m}{n}\ln 2} = (2^{-\ln 2})^{\frac{m}{n}} \approx 0.6185^{\frac{m}{n}}$$

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- We will show that this is not the case: only minor improvements in space are possible

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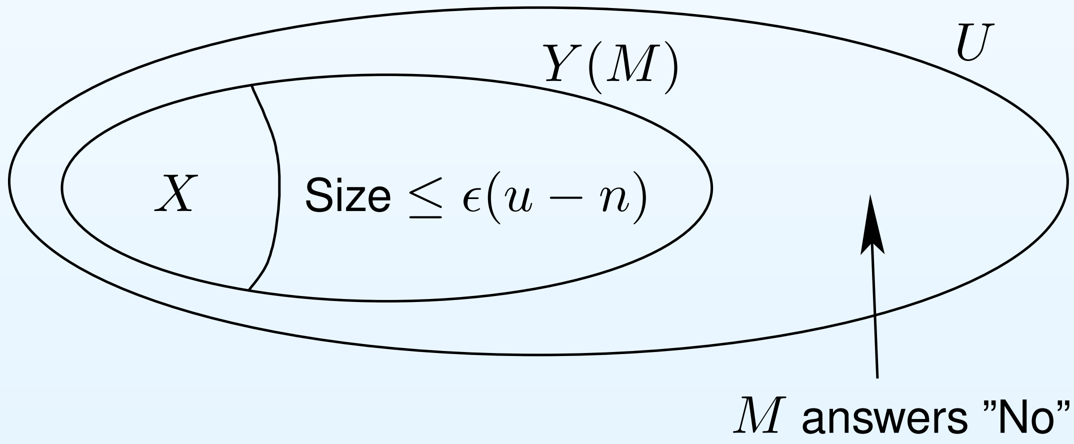

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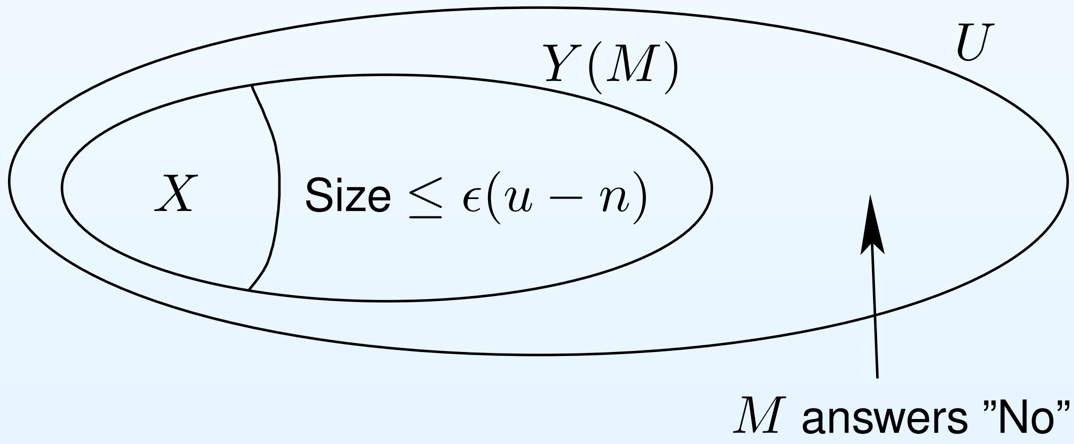

- It follows that $|Y(M)| \leq n + \epsilon(u-n)$

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- However, it needs to represent *all* of the $\binom{u}{n}$ sets X so

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- Similarly $\binom{u}{n} \approx \frac{u^n}{n!}$ since $n \ll \epsilon u \leq u$

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- More complicated data structures with better space bounds exist, for instance compressed Bloom filters

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- Fortunately, Bloom filters work well using much more practical hash functions with weaker guarantees