Massively Parallel Computation

- Computational Model
- Summing
- Sorting
- Minimum Spanning Tree

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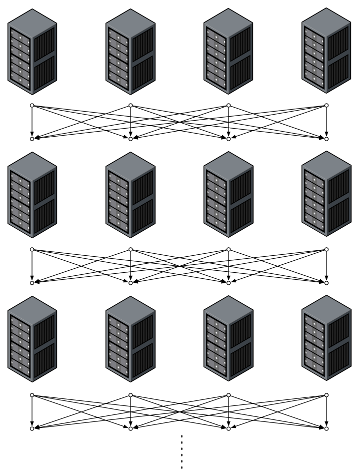
Computational Model

- Massively Parallel Computation (MPC) model.
  - P processors each with space S.
  - Typically $S = N^{\varepsilon}$ and $P = N^{1-\varepsilon}$.
  - Synchronous computation in rounds.
  - Round = local computation + communication.
  - Communication into a processor is < S.
- Complexity model.
  - Rounds and space ($\Longrightarrow$ communication)
  - Computation is free (!)
- Implementations.
  - Map Reduce
  - Bulk-Synchronous parallel

N = problem size

P = number of processors

S = space on each processor



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Summing

7, 42, 3,1, 18, 2, 9, 10, 11, 4, 51, 6, 3, 24, 92, 56, 19, 8, 5, 22, 33

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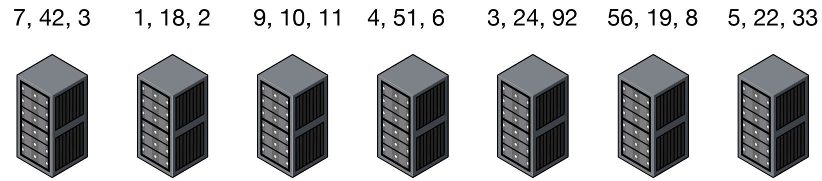
- Sum. Given a list of N integers $A_0, A_1, \dots, A_{N-1}$ compute their sum.
- Input distributed arbitrarily among processors.

Summing



- Assume $S = \Theta\left(\sqrt{N}\right)$ and $P = \Theta\left(\sqrt{N}\right)$
- Sum.
  - Each processor computes local sum and sends to processor 0.
  - Compute global sum at processor 0.
- Rounds. 2.

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Sorting

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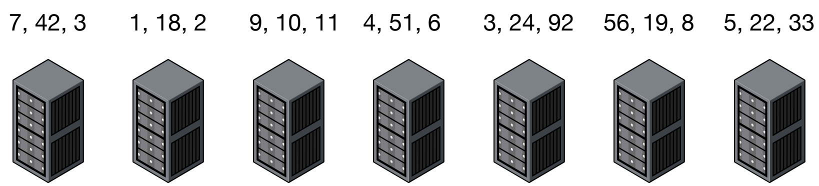
(7,8), (42,18), (3,3), (1,1), (18,13),(2,2), (9,10), (10, 11), (11,12),(4,5),(51,19), (6,7), (3,4), (24,16), (92,21), (56, 20), (19,14), (8,9), (5, 6), (22,15), (33, 17)

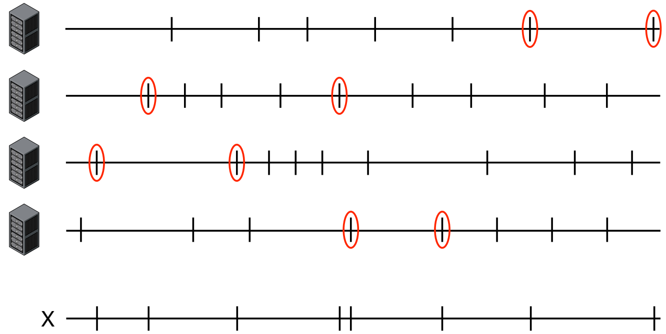
- Sorting. Given a list of N integers $A_0, A_1, \dots, A_{N-1}$ compute list $(A_0, \text{rank}(A_0)), (A_1, \text{rank}(A_1)), \dots, (A_{N-1}, \text{rank}(A_{N-1}))$
- Input and output distributed arbitrarily among processors.

Sorting



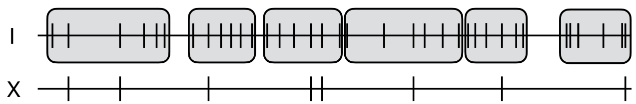
- Goal. Sorting in O(1) rounds whp. with $S = \tilde{\Theta}\left(\sqrt{N}\right)$ and $P = \tilde{\Theta}\left(\sqrt{N}\right)$.
- Idea.
  - Sample $\tilde{\Theta}(\sqrt{N})$ items and use sample to partition items into $\tilde{\Theta}(\sqrt{N})$ ranges.
  - Distribute items according to ranges and sort each range locally.

Sorting



- Sample.
  - Each processor samples its items with probability $2P \ln N / N$ and sends these to processor 0.
  - Processor 0 broadcasts the set of samples to all processors.
  - Let X be the set of samples. $|X| \leq 4P \ln N$ whp.

Sorting



- Lemma. Let I be the sorted input. Consider a partition of I into P ranges of N/P consecutive items. Then, all ranges contain at least one item from X whp.
- Proof.

Pr we don't sample item

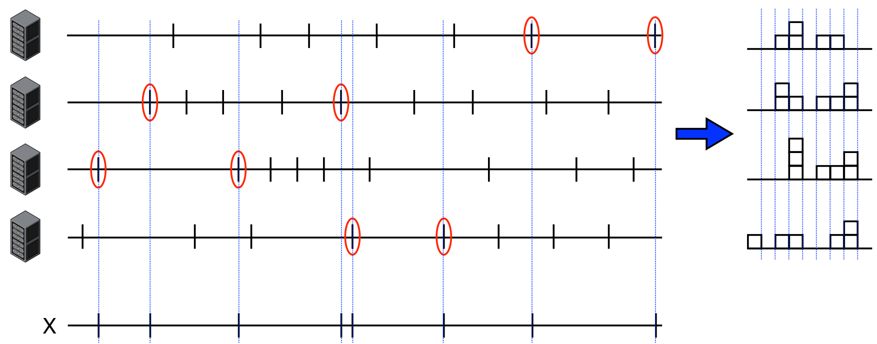
$$\Pr(\text{range contains no items}) = \left(1 - \frac{2P\ln N}{N}\right)^{\frac{N}{P}} \leq e^{-2\ln N} = \frac{1}{N^2}$$

Pr we don't sample item in range

$(1+x)^r \leq e^{rx}$

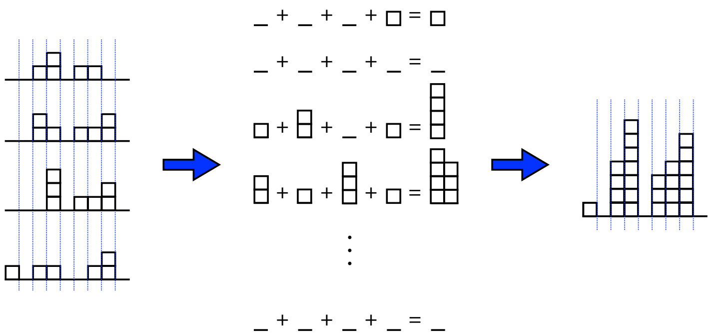
- $\implies \Pr(\text{some range contain no items}) = P \cdot \frac{1}{N^2} < \frac{1}{N}$
- $\implies \Pr(\text{all ranges contain at least 1 item}) > 1 - \frac{1}{N}$

Sorting



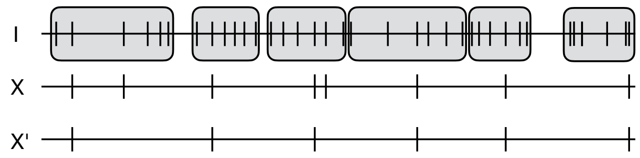
- Compute local histogram.
  - Each processor counts number of items in ranges defined by X.
  - Each histogram uses O(|X|) space.

Sorting



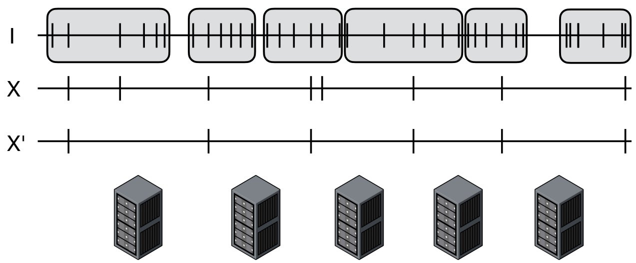
- Compute global histogram.
  - Each processor sends count for range i to processor i mod P.
  - Processor i sums counts for range i mod P and sends sum to processor 0.
  - Processor 0 constructs global count.
  - Each processor is responsible for counting $|X|/P = O(\log N)$ ranges and receives $O(P \log N)$ integers.

Sorting



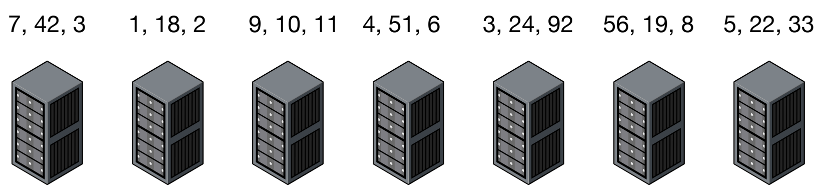
- Select.
  - Processor 0 selects $X' \subseteq X$ such that each range defined by X' contains O(N/P) items from I and |X'| = O(P).
  - Processor 0 broadcasts X' to all machines.
  - Sampling lemma $\Longrightarrow$ X' exists whp.

Sorting



- Exchange.
  - Assign each range defined by X' to a processor.
  - Each processor sends each of its items to processor assigned to corresponding range.
  - Each processor locally sorts its items.
  - Output sorted sequence.

Sorting



- Theorem. Sorting in O(1) rounds whp. with $S = \tilde{\Theta}\left(\sqrt{N}\right)$ and $P = \tilde{\Theta}\left(\sqrt{N}\right)$.

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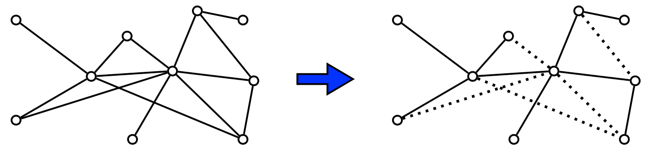
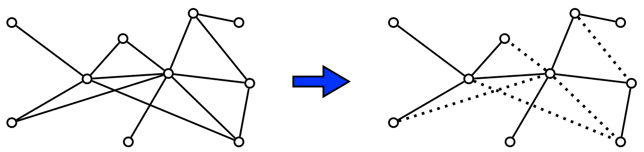
Minimum Spanning Tree



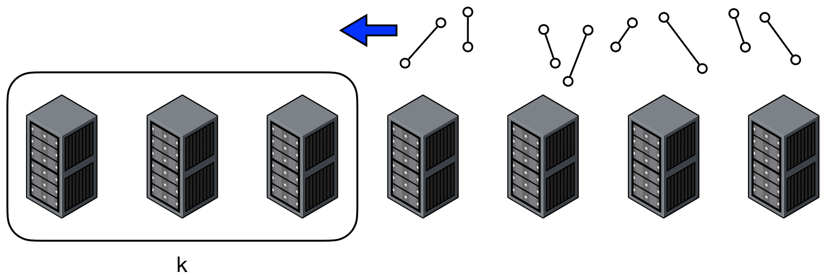
- Minimum spanning tree. Given a connected, weighted, undirected graph compute the minimum spanning tree (MST).
- Input given as list of edges with weights. Output edges in MST.
- Input and output distributed arbitrarily among processors.

Minimum Spanning Tree



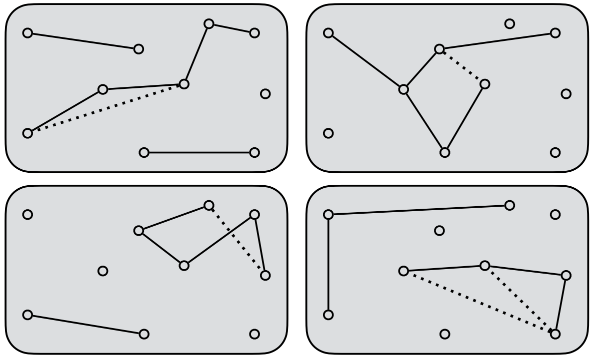
- Let G be graph with n nodes and m edges.
- Goal. MST in $O(1/\varepsilon)$ rounds whp. for $S = \Theta(n^{1+\varepsilon})$ and $P = \Theta(m/S) = \Theta(m/n^{1+\varepsilon})$
- Idea.
  - Repeatly filter edges not part of MST in rounds.
  - When all edges fit on one processor compute the MST directly.

Minimum Spanning Tree



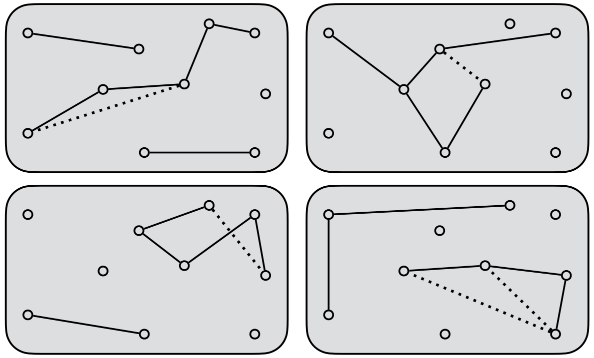
- Shuffle.
  - Let m' be the current edges. Initially, m' = m.
  - Choose $k = 2m'/n^{1+\varepsilon}$ active processors.
  - Distribute edges among active processors randomly.
  - Let E_i be the edges at processor i. $|E_i| = n^{1+\varepsilon}$ whp.

Minimum Spanning Tree



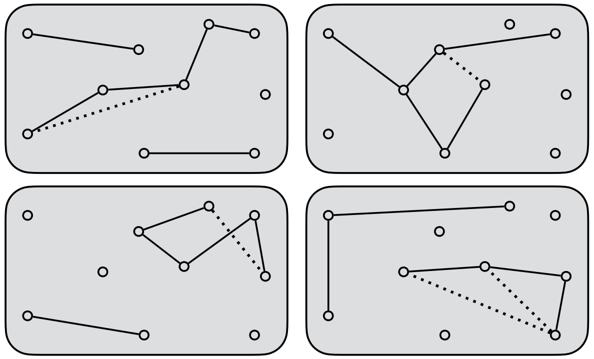
- Filter. Active processor i:
  - computes a local minimum spanning forest of $G = (V, E_i)$.
  - discards all other edges in E_i

Minimum Spanning Tree



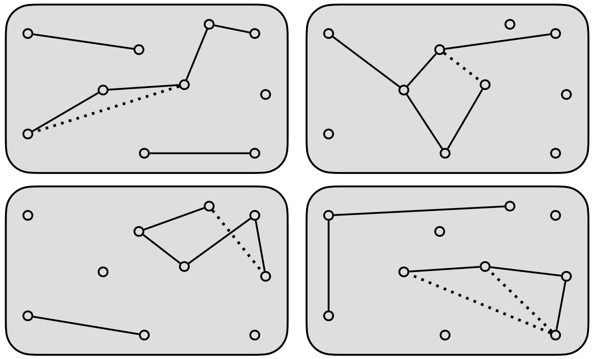
- Repeat.
  - Repeat shuffle and filter step until remaining edges fit on a single machine.
  - Then compute MST.

Minimum Spanning Tree



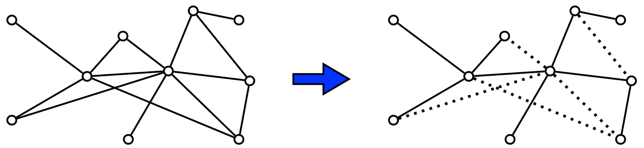
- Correctness.
  - Edges in E_i that are not in the local minimum spanning forest are not in the MST.

Minimum Spanning Tree



- Rounds.
  - Total edges remaining after a round is $\leq k(n-1) = \frac{2m'}{n^{1+\varepsilon}}(n-1) < \frac{2m'}{n^{\varepsilon}}$
  - $\implies$ A round reduces edges by factor n^{ε}
  - $\implies$ After $O(1/\varepsilon)$ rounds the remaining edges is $< n^{1+\varepsilon}$.

Minimum Spanning Tree



- Theorem. MST in $O(1/\varepsilon)$ rounds whp. for $S = \Theta(n^{1+\varepsilon})$ and $P = \Theta(m/n^{1+\varepsilon})$

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