Range Minimum Queries and Lowest Common Ancestor

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Range Minimum Queries and Lowest Common Ancestor

- Range Minimum Queries (RMQ) and Lowest Common Ancestor (LCA)
- RMQ
    - Simple solutions
    - Better solution
    - 2-level solution
- Reduction between RMQ and LCA
- Dynamic RMQ

Range Minimum Queries

- Range minimum query problem. Preprocess array A[1…n] of integers to support
  - RMQ(i,j): return the (entry of) minimum element in A[i…j].

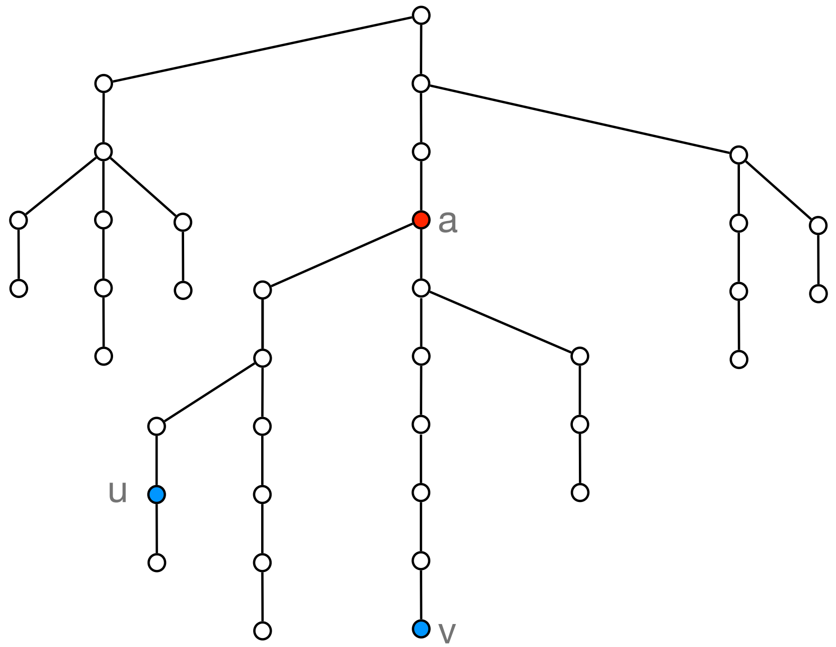
0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

  - RMQ(2,5) = 2 (index 4)

- Basic (extreme) solutions
  - Linear search:
    - Space: O(n). Only keep array (no extra space)
    - Time: O(j-i) = O(n)
  - Save all possible answers: Precompute and save all answers in a table.
    - Space: $O(n^2)$ pairs => $O(n^2)$ space
    - Time: O(1)

Lowest Common Ancestor

- Lowest common ancestor problem. Preprocess rooted tree T with n nodes to support
  - LCA(u,v): return the lowest common ancestor of u and v.



LCA(u,v) = a

Lowest Common Ancestor

- Basic (extreme) solutions
    - Linear search: Follow paths to root and mark when you visit a node.
        - Space: O(n). Only keep tree (no extra space)
        - Time: O(depth of tree) = O(n)
    - Save all possible answers: Precompute and save all answers in a table.
        - Space: $O(n^2)$ pairs => $O(n^2)$ space
        - Time: O(1)

RMQ and LCA

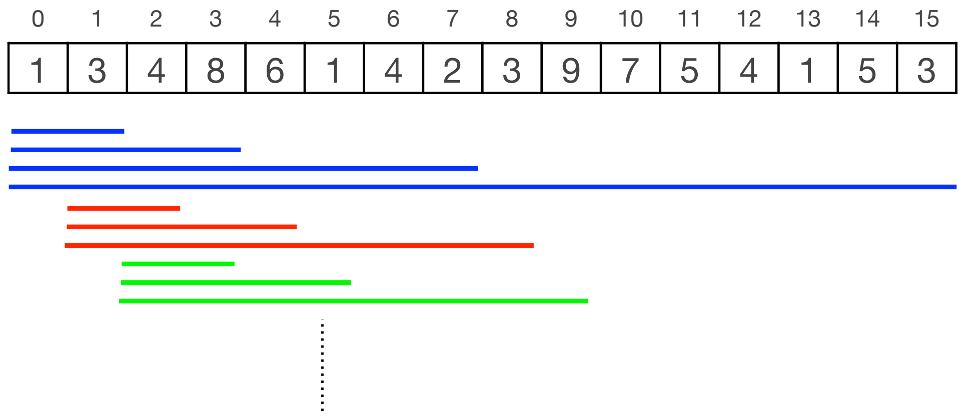
- Outline.
  - Can solve both RMQ and LCA in linear space and constant time.
    - First solution to RMQ
    - Solution to a special case of RMQ.
    - See that RMQ and LCA are equivalent (can reduce one to the other both ways).

RMQ

Sparse table solution

RMQ: Sparse table solution

- Save the result for all intervals of length a power of 2.

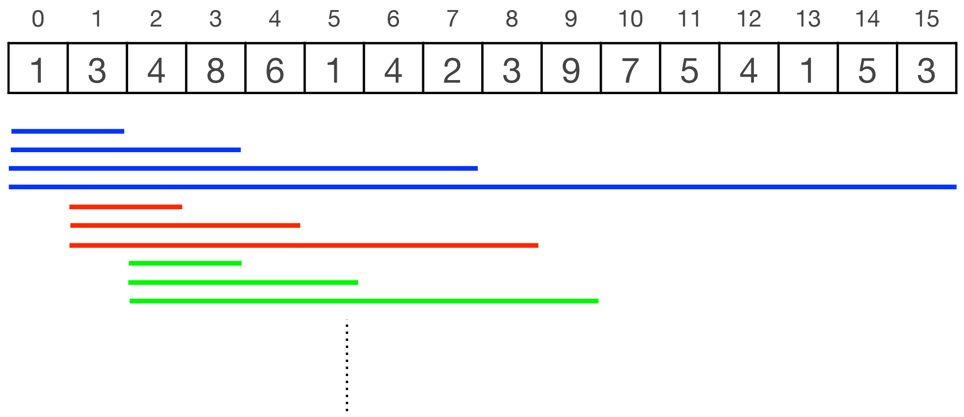


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	4	8	6	1	4	2	3	9	7	5	4	1	5	3

	0	1	2	3	4
0	1	1	1	1	1
1	3	3	3	1	
2	4	4	1	1	
3	8	6	1	1	
4	6	1	1	1	
5	1	1	1	1	
6	4	2	2	1	
7	2	2	2	1	
8	3	3	3	1	
9	9	7	4		
10	7	5	1		
11	5	4	1		
12	4	1	1		
13	1	1			
14	5	3			
15	3				

RMQ: Sparse table solution

- Save the result for all intervals of length a power of 2.

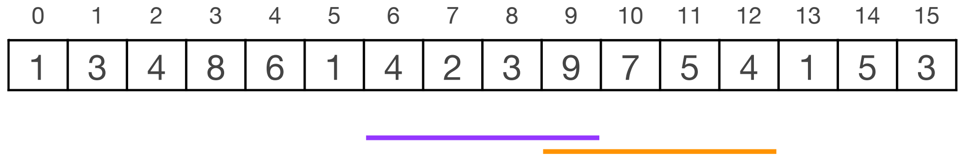


	0	1	2	3	4
0	1	1	1	1	1
1	3	3	3	1	
2	4	4	1	1	
3	8	6	1	1	
4	6	1	1	1	
5	1	1	1	1	
6	4	2	2	1	
7	2	2	2	1	
8	3	3	3	1	
9	9	7	4		
10	7	5	1		
11	5	4	1		
12	4	1	1		
13	1	1			
14	5	3			
15	3				

- Space: O(n log n)

RMQ: Sparse table solution

- Save the result for all intervals of length a power of 2.



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	4	8	6	1	4	2	3	9	7	5	4	1	5	3

RMQ(6, 12) = ?

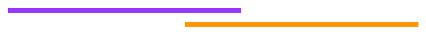
	0	1	2	3	4
0	1	1	1	1	1
1	3	3	3	1	
2	4	4	1	1	
3	8	6	1	1	
4	6	1	1	1	
5	1	1	1	1	
6	4	2	2	1	
7	2	2	2	1	
8	3	3	3	1	
9	9	7	4		
10	7	5	1		
11	5	4	1		
12	4	1	1		
13	1	1			
14	5	3			
15	3				

- Space: O(n log n)

RMQ: Sparse table solution

- Save the result for all intervals of length a power of 2.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	4	8	6	1	4	2	3	9	7	5	4	1	5	3



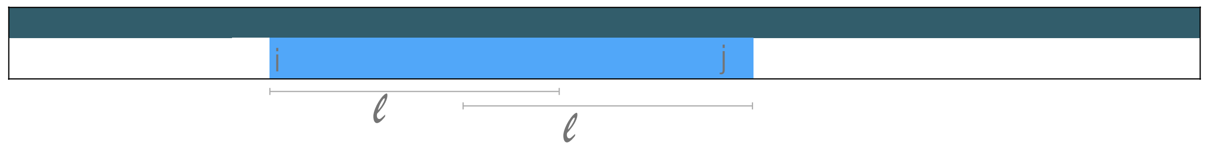
RMQ(6, 12) = min(RMQ(6,9), RMQ(9,12)) = min(2,4) = 2

	0	1	2	3	4
0	1	1	1	1	1
1	3	3	3	1	
2	4	4	1	1	
3	8	6	1	1	
4	6	1	1	1	
5	1	1	1	1	
6	4	2	2	1	
7	2	2	2	1	
8	3	3	3	1	
9	9	7	4		
10	7	5	1		
11	5	4	1		
12	4	1	1		
13	1	1			
14	5	3			
15	3				

- Space: O(n log n)

RMQ: Sparse table solution

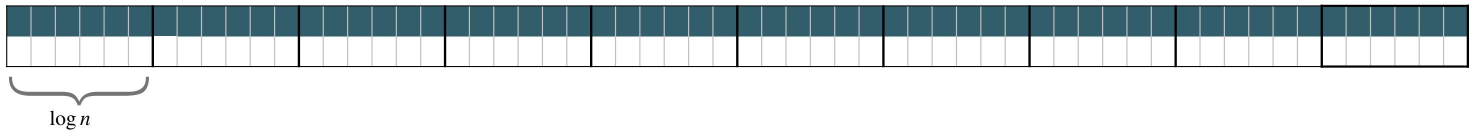
- Query:



  - Any interval is the union of two power of 2 intervals.
    - k largest number such that $2^k \leq j - i + 1$.
  - Lookup results for the two intervals and take minimum.
- Time: O(1)
- Space: O(n log n)
- Preprocesing time: O(n log n)
  - To compute results for length 2^i use results for length 2^{i-1}.

Reducing Space

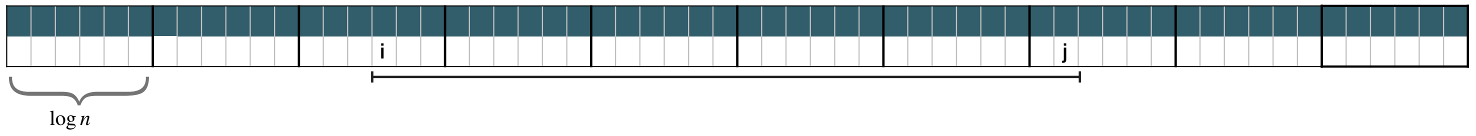
- Divide A into blocks of size $\log n$



- 2-level data structure:
  - A data structure on minimums of the blocks
  - A data structure for queries inside blocks.

Reducing Space

- Divide A into blocks of size $\log n$



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Reducing Space

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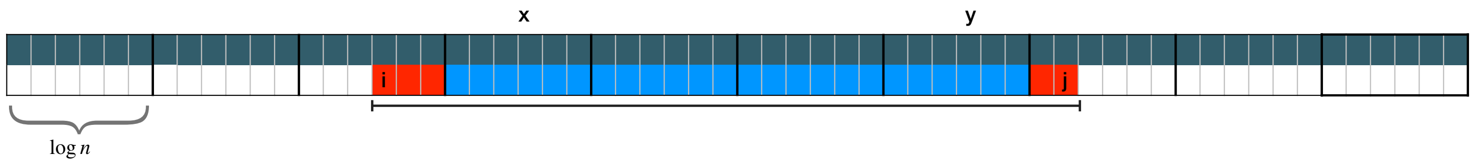
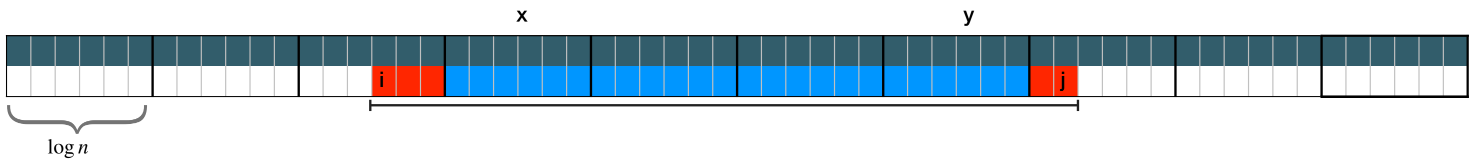


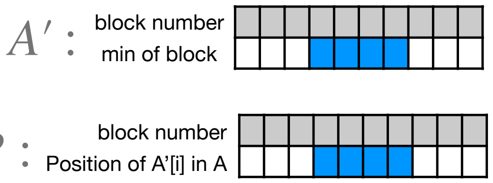
- 2-level data structure:
  - A data structure on minimums of the blocks
  - A data structure for queries inside blocks.

- RMQ(i,j) = min{ RMQ on blocks x to y,
  RMQ inside block x-1,
  RMQ inside block y+1 }.

Reducing Space: Data Structure on Blocks

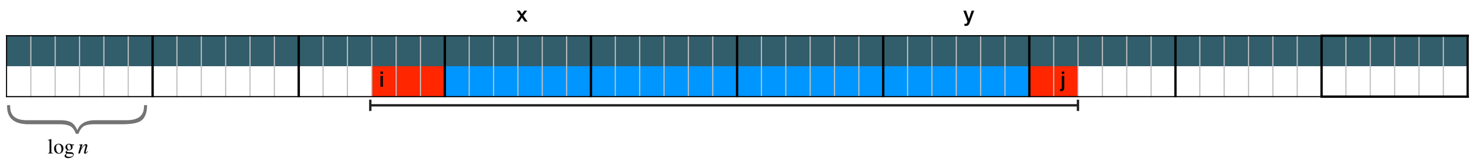


- Two new arrays.
  - Array A': minimum from each block
  - P: position in A where $A'[i]$ occurs.

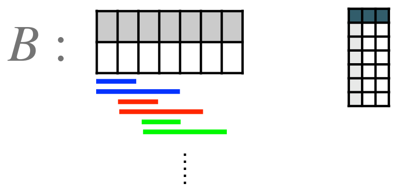


- Sparse table data structure on A'.
- Space: $O(|A'|\log|A'|) = O\left(\frac{n}{\log n}\cdot\log\frac{n}{\log n}\right) = O(n)$.
- Time: $O(1)$

Reducing Space: Data Structure Inside Blocks

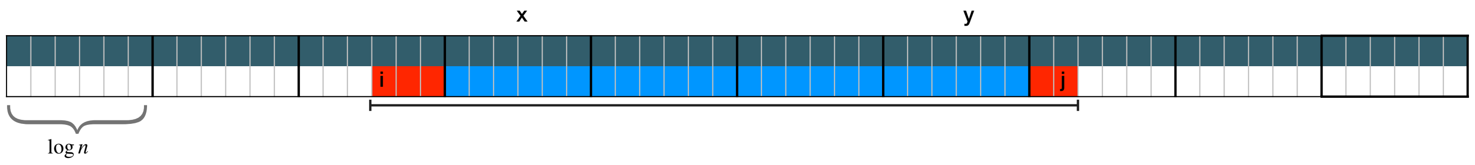


- For each block B:
  - Sparse table
- Sparse table data structure on B.



- Space: $O(|B|\log|B|) = O\left(\frac{\log n}{\log\log n}\cdot\log\frac{\log n}{\log\log n}\right) = O(\log n\cdot\log\log n)$.
- Time: $O(1)$
- Total space for all blocks: $O\left(\frac{n}{\log n}\cdot\log n\cdot\log\log n\right) = O(n\log\log n)$.

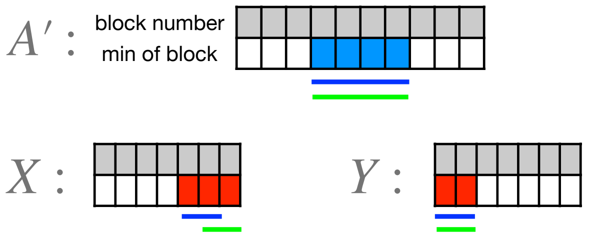
Reducing Space: Total Space



- Sparse table on minimum of blocks: $O(n)$
- Sparse table on each block: $O(n \log \log n)$
- Gives solution using
  - Space: $O(n \log \log n)$ space.
  - Time: O(1) + O(1) = O(1).

3 sparse table lookups

min{•,•,•}



±1RMQ

RMQ: Linear space

- Consider ±1RMQ: consecutive entries differ by 1.

0	1	2	3	4	5	6	7	8	9	10	11	12
4	5	6	5	4	3	2	3	2	3	4	5	4

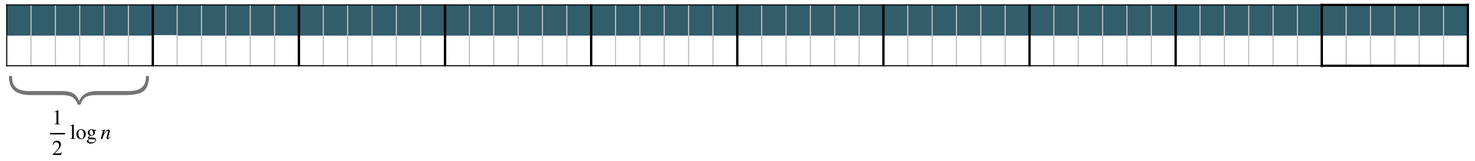
- 2-level solution: Combine
  - $O(n \log n)$ space, $O(1)$ time
  - $O(n^2)$ space, $O(1)$ time.

$$\Downarrow$$

  - $O(n)$ space, $O(1)$ time.

±1RMQ

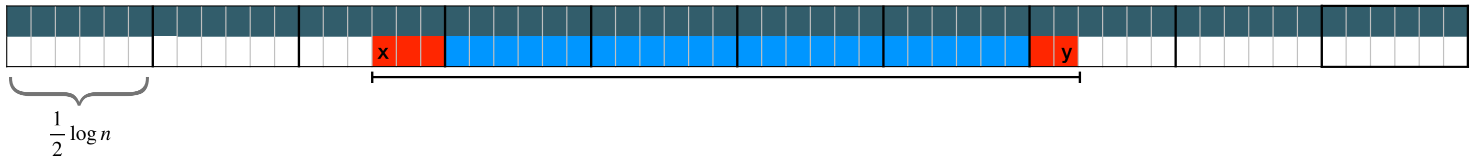
- Divide A into blocks of size $\frac{1}{2}\log n$



- 2-level data structure:
  - Sparse table on blocks
  - Tabulation inside blocks.

±1RMQ

- Divide A into blocks of size $\frac{1}{2}\log n$



- 2-level data structure:
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- RMQ(x,y) = min{ RMQ on blocks i to j,
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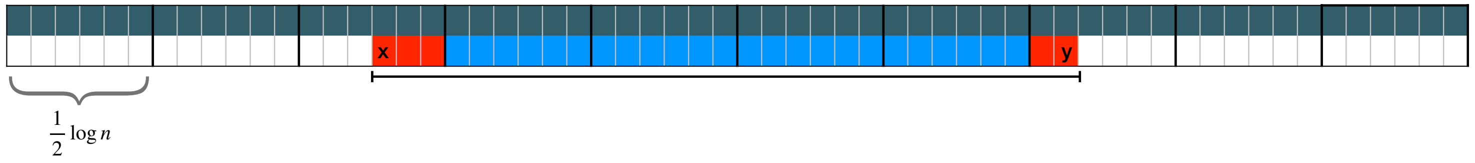
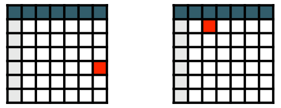
±1RMQ: Data structure inside blocks





- Precompute and save all answers for each block.
- Gives solution using
  - Space: O(n) + space for precomputed tables.
  - Time: O(1) + O(1) + O(1) = O(1).

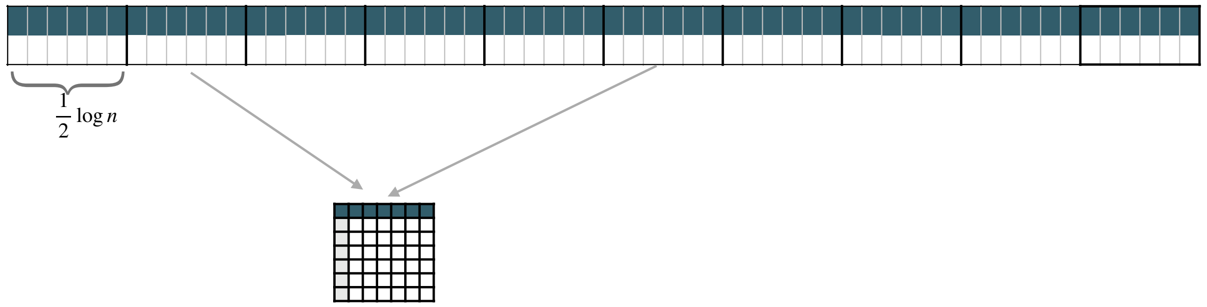
2 table lookups — sparse table — min{•,•,•}

±1RMQ: Storing the precomputed tables

- Naively: $\log^2 n$ for each table => n log n space. 😕
- Observation: If X[i] = Y[i] + c then all RMQ answers are the same for X and Y.
  - X = [7, 6, 5, 6, 5, 4]
  - Y = [3, 2, 1, 2, 1, 0]
- Describe block by sequence of +1s and -1s:
  - X = Y = -1, -1, +1, -1, -1.
- How many different block descriptions are there?
  - length of sequence = $\frac{1}{2}\log n - 1$
  - #sequences = $2^{\frac{1}{2}\log n - 1} \leq \sqrt{n}$.

±1RMQ: Data structure inside blocks

- Precompute and save all answers for each normalized block.
- Size of a table: O($\log^2$ n)
- For each block save which precomputed table it uses.



- Space: $O(\sqrt{n}\cdot\log^2 n) + O(n/\log n) = O(n)$
- Plugging into 2-level solution:
  - Space: O(n) + space for precomputed tables = O(n).

LCA and RMQ

RMQ and LCA

- We will show

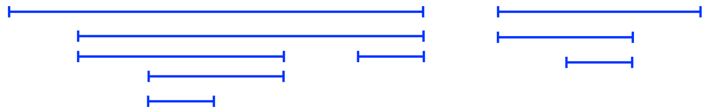
$$\begin{tikzcd}\text{RMQ} \arrow[r, "\text{reduces to}"'] & \text{LCA} \arrow[r, "\text{reduces to}"'] & \pm 1\text{RMQ}\end{tikzcd}$$

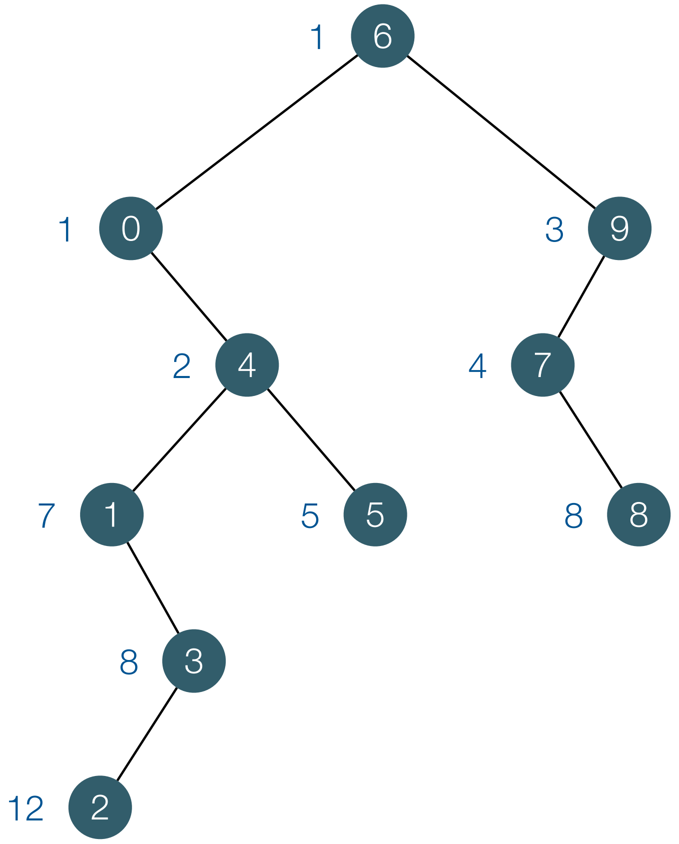
If there is a solution to LCA using s(n) space and t(n) time, then there is a solution to RMQ using O(s(n)) space and O(t(n)) time.

If there is a solution to ±1RMQ using s(n) space and t(n) time, then there is a solution to LCA using O(s(n)) space and O(t(n)) time.

RMQ to LCA: Cartesian Tree

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

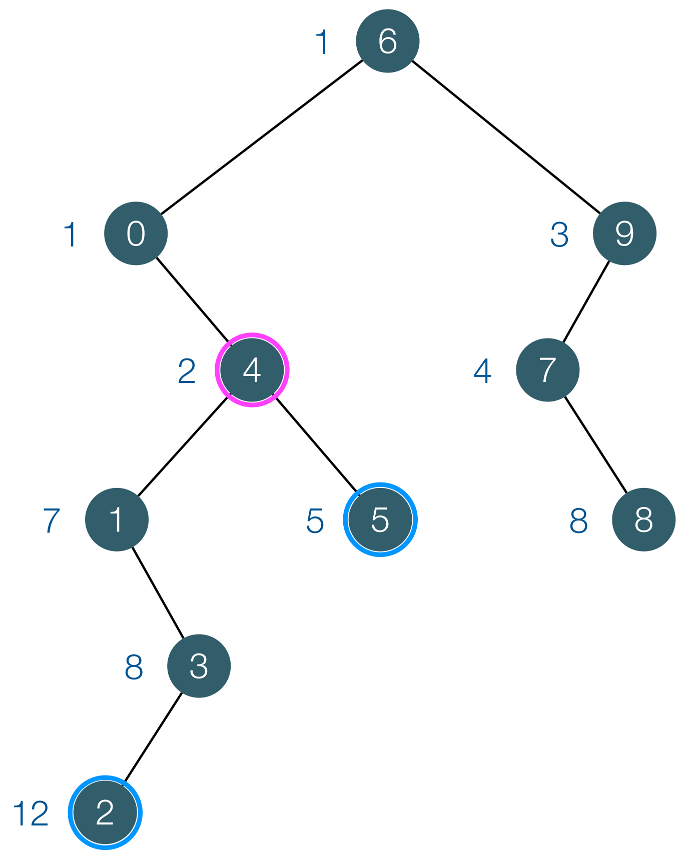




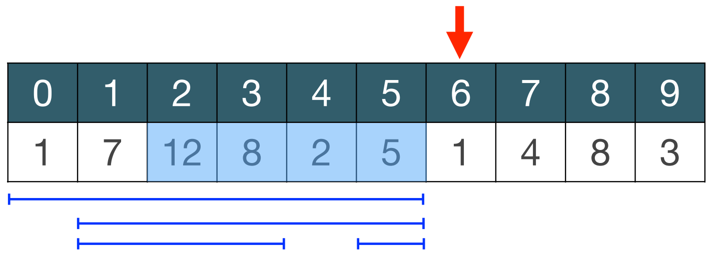
RMQ to LCA: Cartesian Tree

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

- RMQ(2,5)

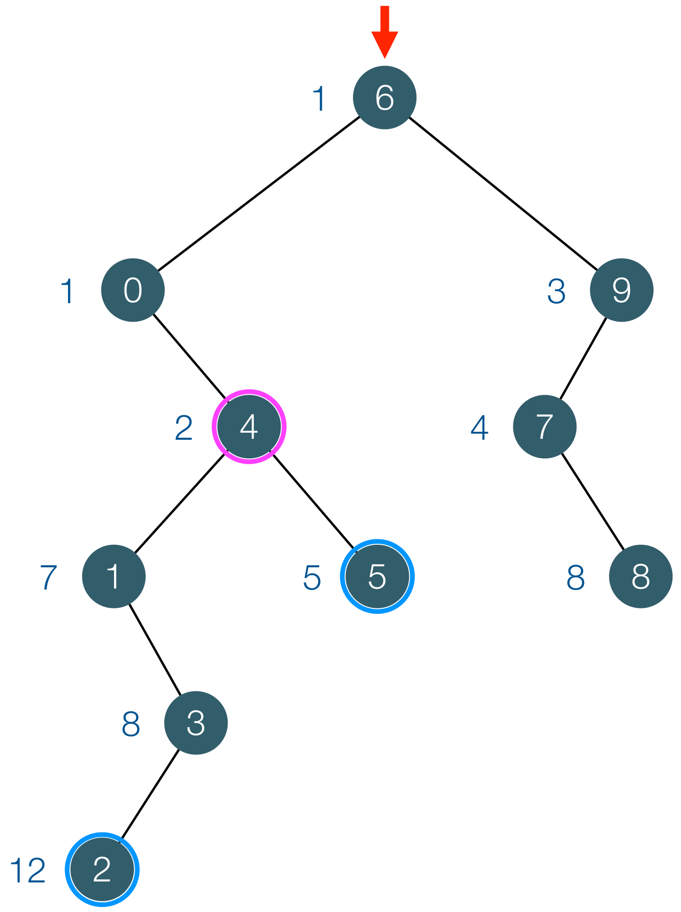


RMQ to LCA: Cartesian Tree

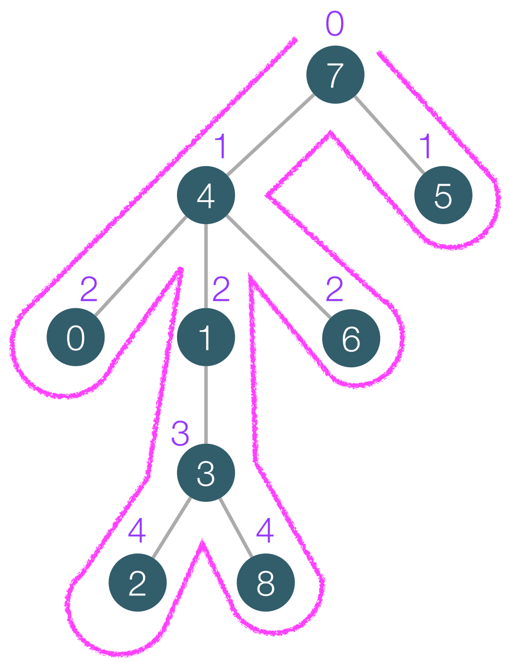


0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

- RMQ(2,5) = LCA(2,5)



LCA to ±1RMQ



- E =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	4	0	4	1	3	2	3	8	3	1	4	6	4	7	5	7

- A =

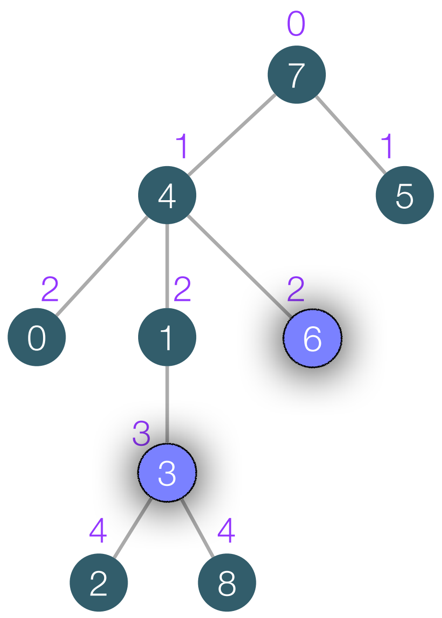
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	2	1	2	3	4	3	4	3	2	1	2	1	0	1	0

- R =

0	1	2	3	4	5	6	7	8
2	4	6	5	1	15	12	0	8

- E: Euler tour representation. preorder walk, write id of node when met.
- A: depth of node node in E[i].
- R: first occurrence in E of node with id i
- LCA(i, j) = E[RMQ_A(R[i], R[j])].

LCA to ±1RMQ



- E =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	4	0	4	1	3	2	3	8	3	1	4	6	4	7	5	7

- A =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	2	1	2	3	4	3	4	3	2	1	2	1	0	1	0

- R =

0	1	2	3	4	5	6	7	8
2	4	6	5	1	15	12	0	8

- E: Euler tour representation. preorder walk, write id of node when met.
- A: depth of node node in E[i].
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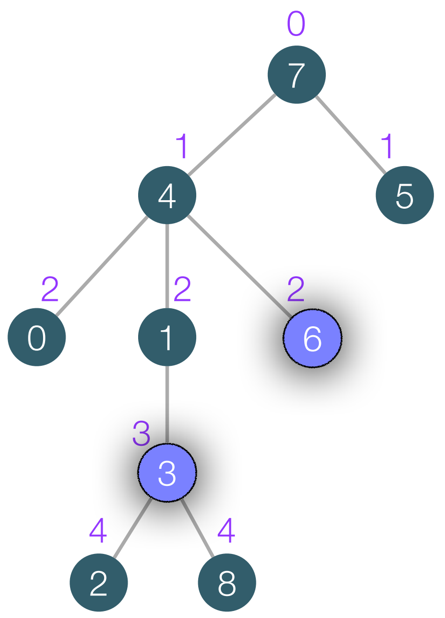
LCA to ±1RMQ



- E =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	4	0	4	1	3	2	3	8	3	1	4	6	4	7	5	7

- A =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	2	1	2	3	4	3	4	3	2	1	2	1	0	1	0

- R =

0	1	2	3	4	5	6	7	8
2	4	6	5	1	15	12	0	8

- E: Euler tour representation. preorder walk, write id of node when met.
- A: depth of node node in E[i].
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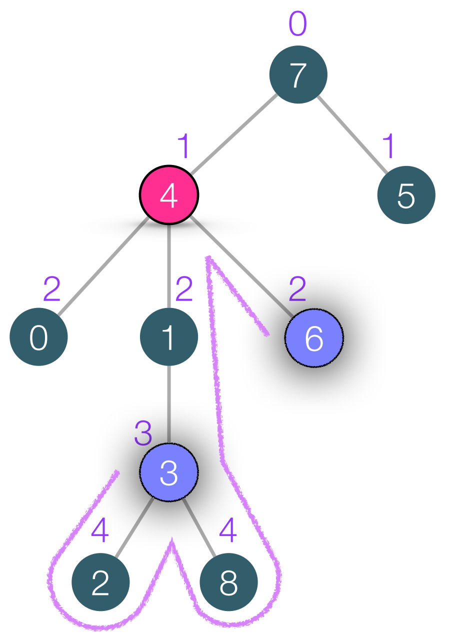
LCA to ±1RMQ



- E =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	4	0	4	1	3	2	3	8	3	1	4	6	4	7	5	7

- A =

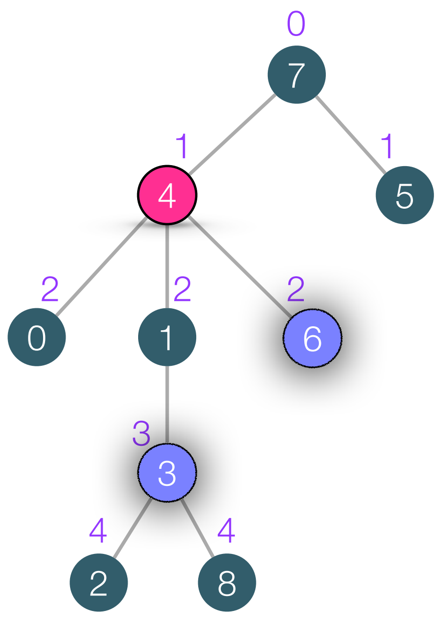
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	2	1	2	3	4	3	4	3	2	1	2	1	0	1	0

- R =

0	1	2	3	4	5	6	7	8
2	4	6	5	1	15	12	0	8

- E: Euler tour representation. preorder walk, write id of node when met.
- A: depth of node node in E[i].
- R: first occurrence in E of node with id i
- LCA(i, j) = E[RMQ$_A$(R[i], R[j])].

LCA to ±1RMQ



- E =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	4	0	4	1	3	2	3	8	3	1	4	6	4	7	5	7

|E| = 2n -1

- A =

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	2	1	2	3	4	3	4	3	2	1	2	1	0	1	0

|A| = 2n-1

- R =

0	1	2	3	4	5	6	7	8
2	4	6	5	1	15	12	0	8

|R| = n

Space O(n):

- 3 tables
- ±1RMQ data structure on table of length 2n

- E: Euler tour representation. preorder walk, write id of node when met.
- A: depth of node node in E[i].
- R: first occurrence in E of node with id i
- LCA(i, j) = E[RMQ$_A$(R[i], R[j])].

RMQ to LCA to ±RMQ

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

E

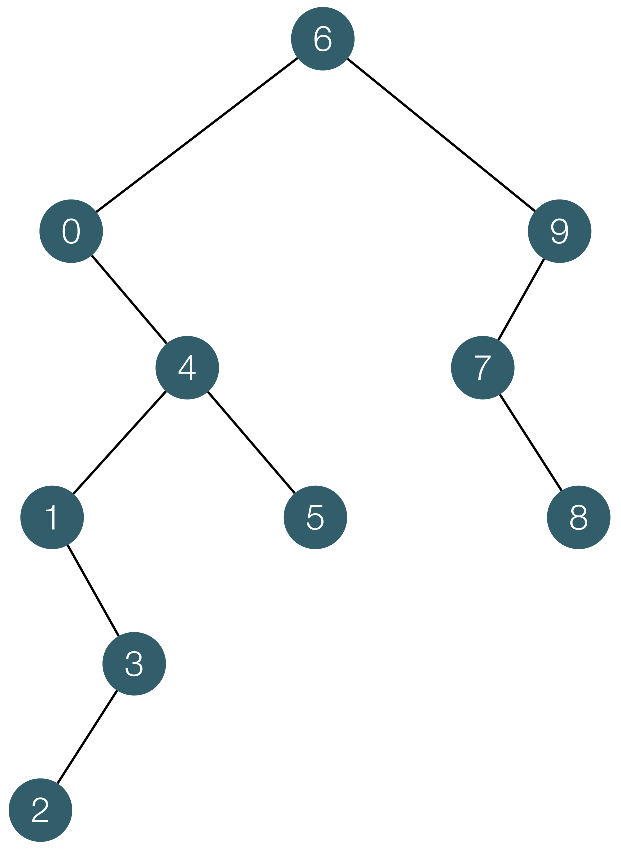
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

R

0	1	2	3	4	5	6	7	8	9



RMQ to LCA: Cartesian Tree

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

E

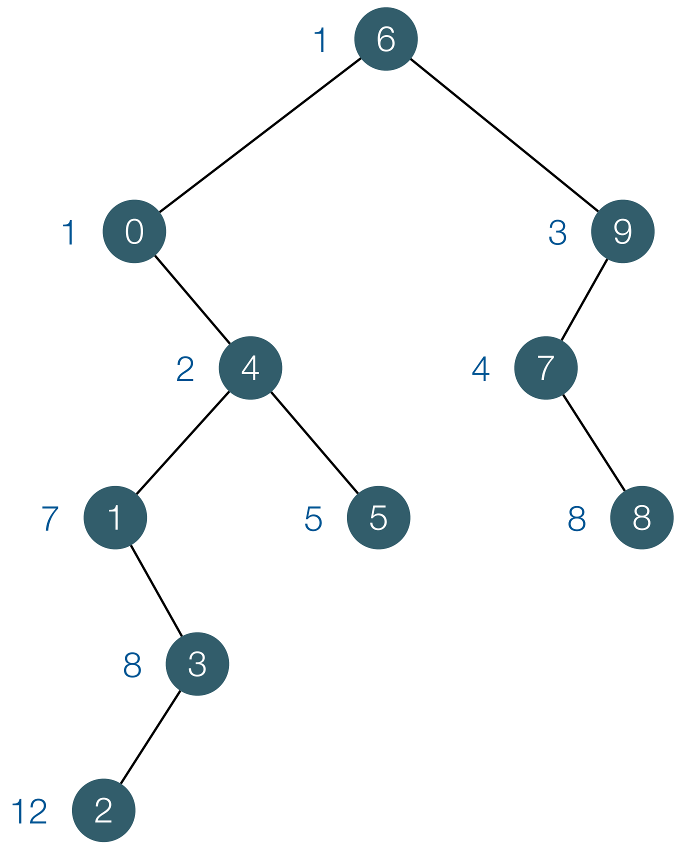
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
6	0	4	1	3	2	3	1	4	5	4	0	6	9	7	8	7	9	6

A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	2	3	4	5	4	3	2	3	2	1	0	1	2	3	2	1	0

R

0	1	2	3	4	5	6	7	8	9
1	3	5	4	2	9	0	14	15	13



RMQ to LCA: Cartesian Tree

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

E

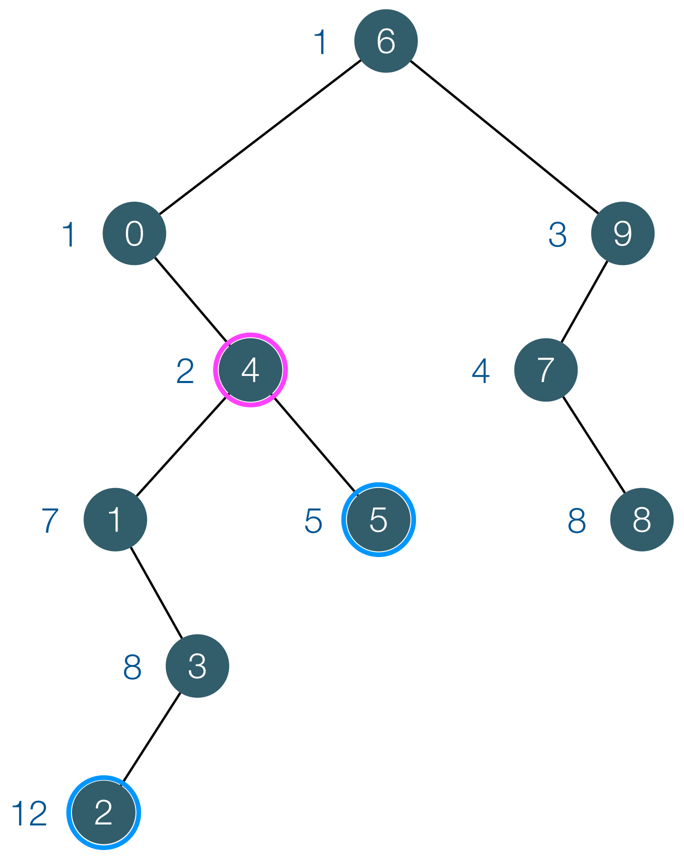
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
6	0	4	1	3	2	3	1	4	5	4	0	6	9	7	8	7	9	6

A

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1	2	3	4	5	4	3	2	3	2	1	0	1	2	3	2	1	0

R

0	1	2	3	4	5	6	7	8	9
1	3	5	4	2	9	0	14	15	13



- RMQ(2,5) = LCA(2,5) = E[RMQ$_A$(R[2], R[5])].

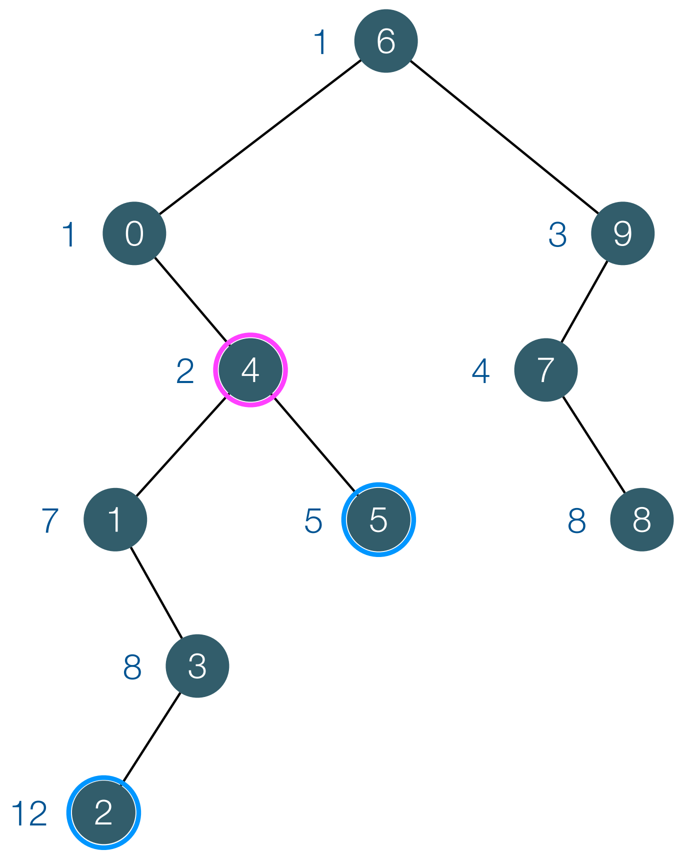
RMQ to LCA: Cartesian Tree

0	1	2	3	4	5	6	7	8	9
1	7	12	8	2	5	1	4	8	3

E	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	6	0	4	1	3	2	3	1	4	5	4	0	6	9	7	8	7	9	6

A	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	0	1	2	3	4	5	4	3	2	3	2	1	0	1	2	3	2	1	0

R	0	1	2	3	4	5	6	7	8	9
	1	3	5	4	2	9	0	14	15	13



- RMQ(2,5) = LCA(2,5) = E[RMQ$_A$(R[2], R[5])].

RMQ and LCA

$$\begin{tikzcd}
\text{RMQ} \arrow[r, "\text{reduces to}"'] & \text{LCA} \arrow[r, "\text{reduces to}"'] & \pm 1\text{RMQ}
\end{tikzcd}$$

If there is a solution to LCA using s(n) space and t(n) time, then there is a solution to RMQ using O(s(n)) space and O(t(n)) time.

If there is a solution to ±1RMQ using s(n) space and t(n) time, then there is a solution to LCA using O(s(n)) space and O(t(n)) time.

- **Theorem.** RMQ and LCA can be solved in O(n) space and O(1) query time.