# Range Reporting

- Range reporting problem
- 1D range reporting
- 2D range reporting
  - Range trees
  - Predecessor in nested sets
  - kD trees

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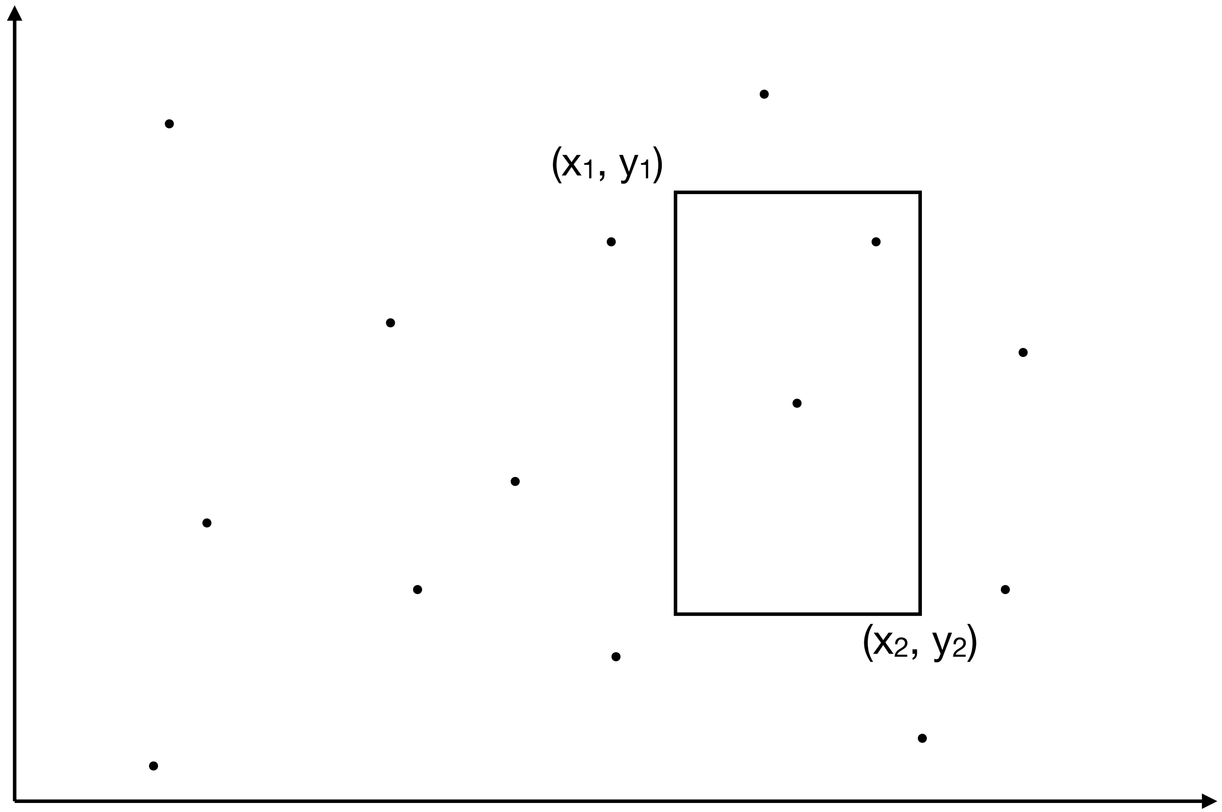
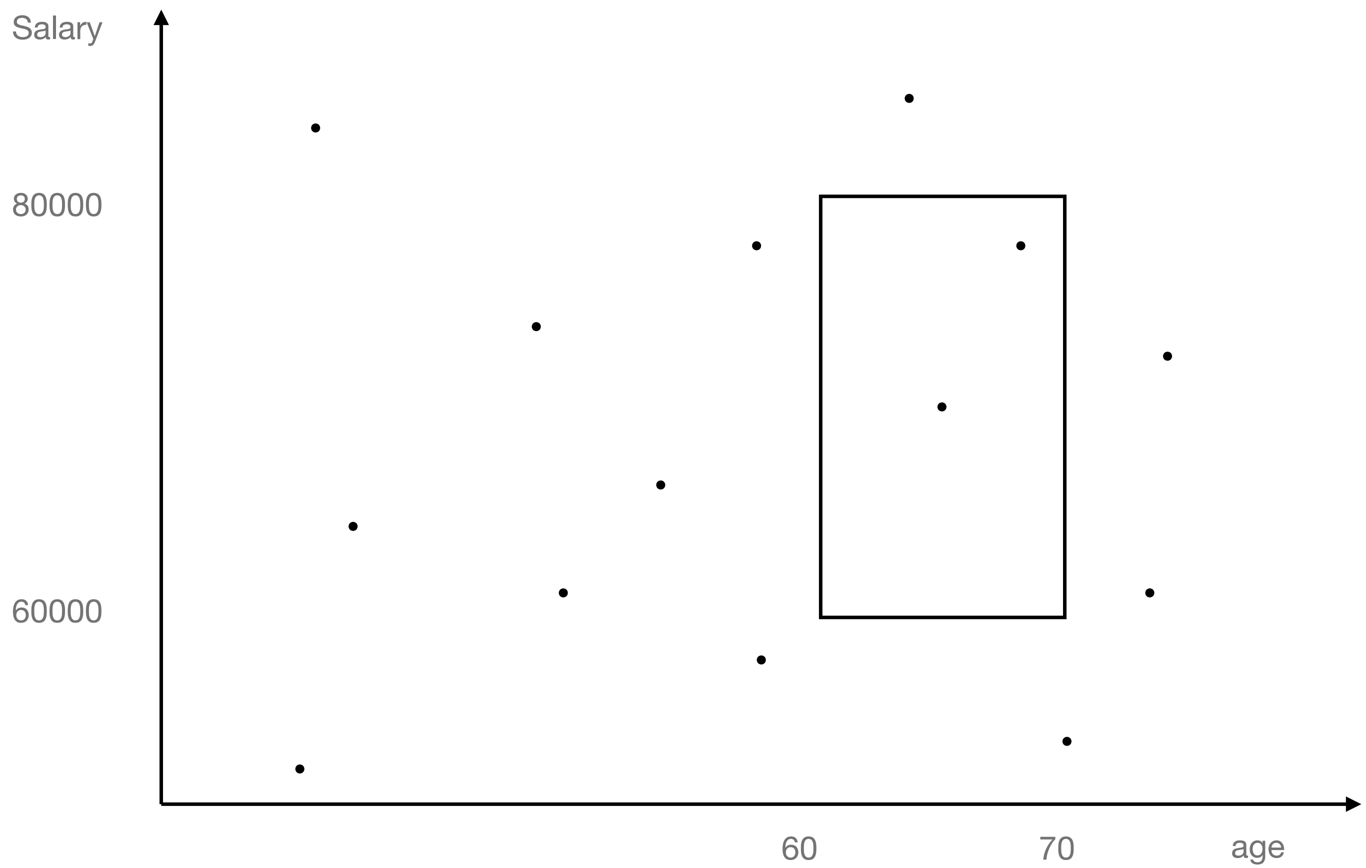
# Range Reporting Problem

- 2D range reporting problem. Preprocess at set of points $P \subseteq \Re^2$ to support
  - report($x_1$, $y_1$, $x_2$, $y_2$): Return the set of points in $R \cap P$, where R is rectangle given by ($x_1$, $y_1$) and ($x_2$, $y_2$).



# Applications

- Relational databases. SELECT all employees between 60 and 70 years old with a montly salary between 60000 and 80000 DKr



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# 1D Range Reporting

- 1D range reporting. Preprocess a set of n points $P \subseteq \Re$ to support:
  - report($x_1$, $x_2$): Return the set of points in interval [$x_1$, $x_2$]
- Output sensitivity. Time should depend on the size of the output.
- Simplifying assumption. Only comparison-based techniques (e.g. no hashing or bittricks).
- Solutions?

# 1D Range Reporting

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 8 | 15 | 17 | 23 | 25 | 26 | 27 | 30 | 46 | 51 | 52 | 65 | 66 | 70 |

- Sorted array. Store P in sorted order.
- Report($x_1$, $x_2$): Binary search for predecessor of $x_1$. Traverse array until $> x_2$.
- Time. O(log n + occ)
- Space. O(n)
- Preprocessing. O(n log n)

# 1D Range Reporting

- Theorem. We can solve the 1D range reporting problem in
  - O(n) space.
  - O(log n + occ) time for queries.
  - O(n log n) preprocessing time.
- Optimal in comparison-based model.

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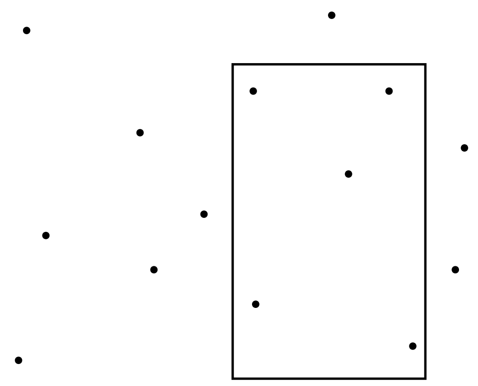
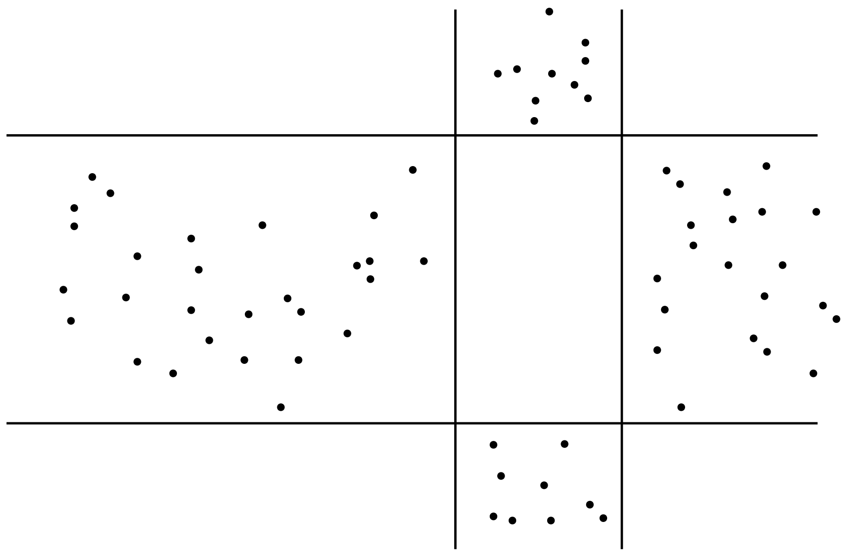
# 2D Range reporting

- Goal. 2D range reporting with
  - O(n log n) space and O(log n + occ) query time or
  - O(n) space and $O(\sqrt{n} + \mathrm{occ})$ query time.
- Solution in 4 steps.
  - Generalized 1D range reporting.
  - 2D range trees.
  - 2D range trees with bridges.
  - kD trees.

# Generalized 1D Range Reporting

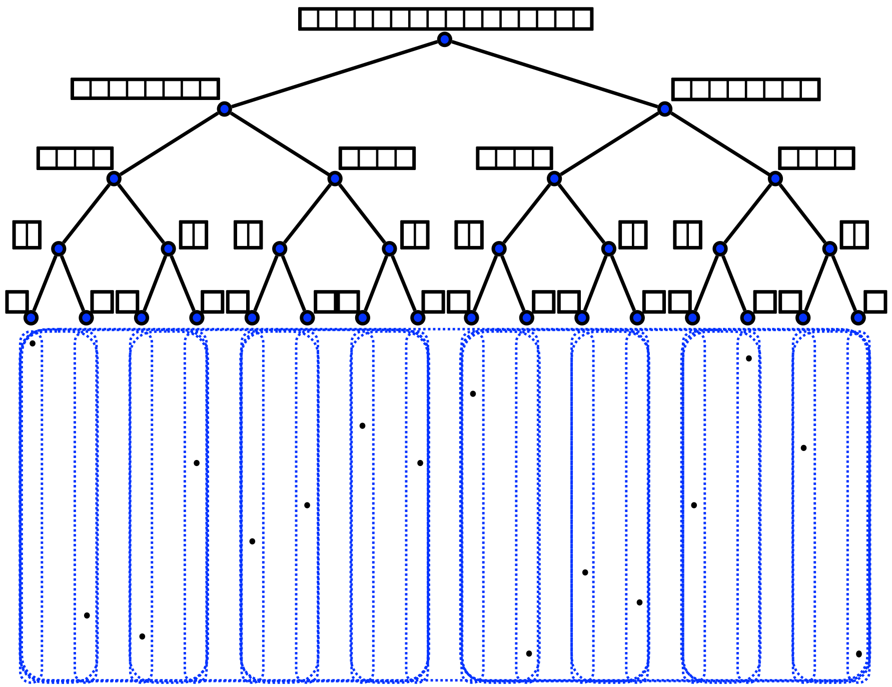
- Data structure.
  - Sorted array over x-coordinate
  - Sorted array over y-coordinate
- Report($x_1$, $y_1$, $x_2$, $y_2$):
  - Compute all points $R_x$ in x-range.
  - Compute all points $R_y$ in y-range.
  - Return $R_x \cap R_y$
- Time?





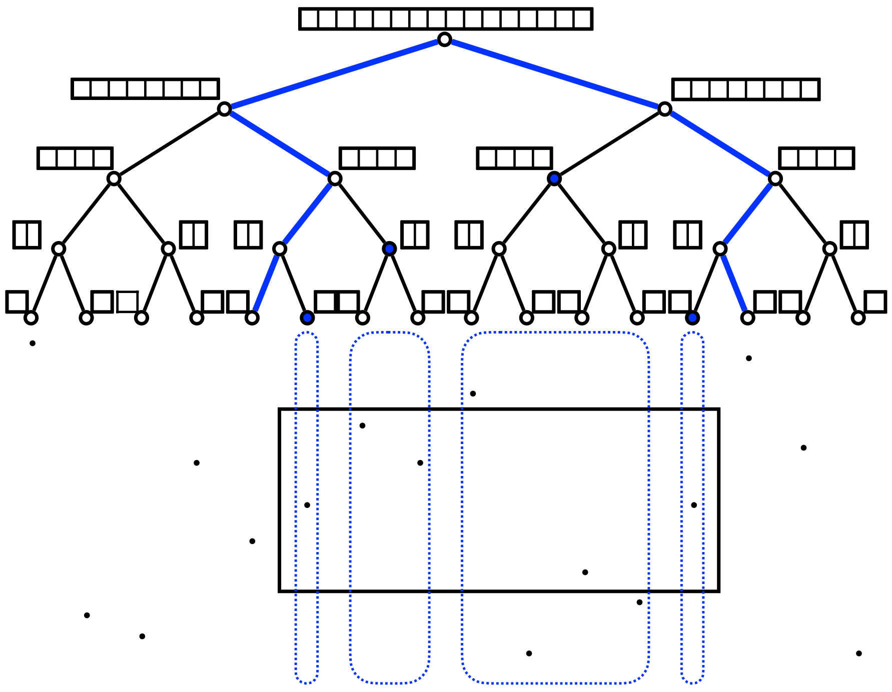
# 2D Range Trees

- Data structure.
  - Perfectly balanced binary tree over x-coordinate.
  - Each node v stores array of points below v sorted by y coordinate.
- Space?
  - O(n log n).
- Preprocessing time. O(n log n)



# 2D Range Trees

- Report($x_1$, $y_1$, $x_2$, $y_2$): Find paths to predecessor of $x_1$ and successor of $x_2$.
  - At each off-path node do 1D query on y-range.
  - Return union of results.
- Time?
  - Predecessor + successor: O(log n)
  - < 2log n 1D queries: O(log n + occ in subrange) time per query.
  - $\implies$ total $O(\log^2 n + occ)$ time.



# 2D Range Reporting

- Theorem. We can solve the 2D range reporting problem in
  - $O(n \log n)$ space.
  - $O(\log^2 n + occ)$ time for queries.
  - $O(n \log n)$ preprocessing time.

- Challenge. Do we really need the $\log^2 n$ term for queries? Can we get (optimal) $O(\log n + occ)$ instead?

# Range Reporting

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  - Predecessor in nested sets
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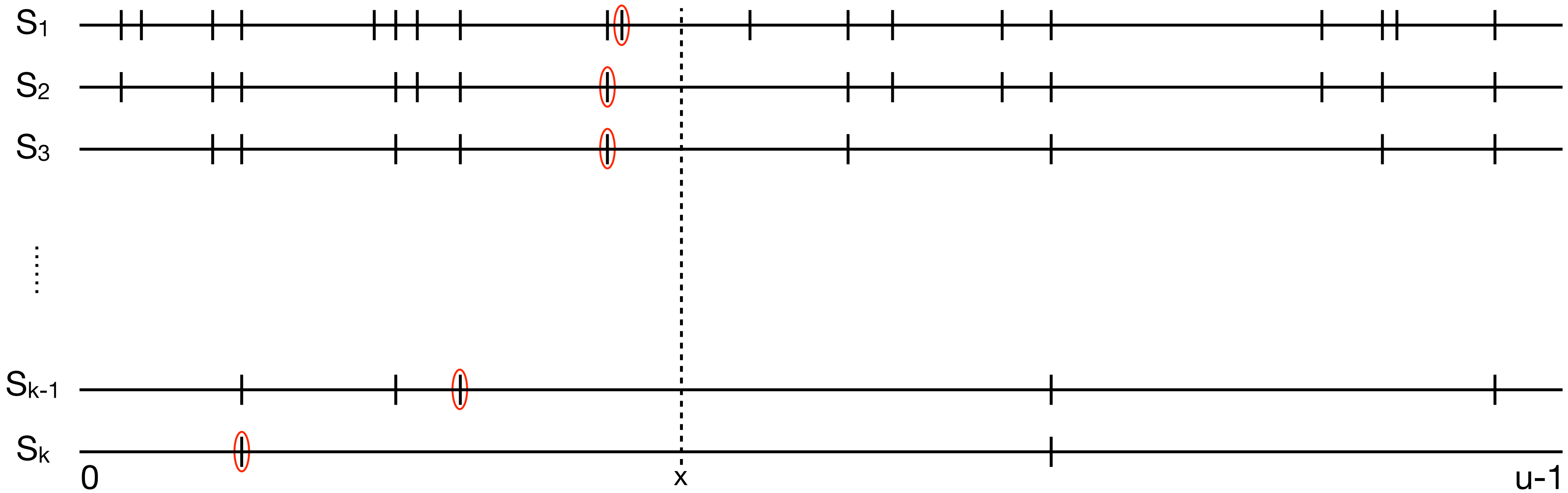
# Predecessor in Nested Sets

- Predecessor problem in nested sets. Let $S = \{S_1, S_2, \ldots, S_k\}$ be a family of sets from universe U such that $U \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_k$.
  - predecessor(x): return the predecessor of x in each of $S_1, S_2, \ldots, S_k$.



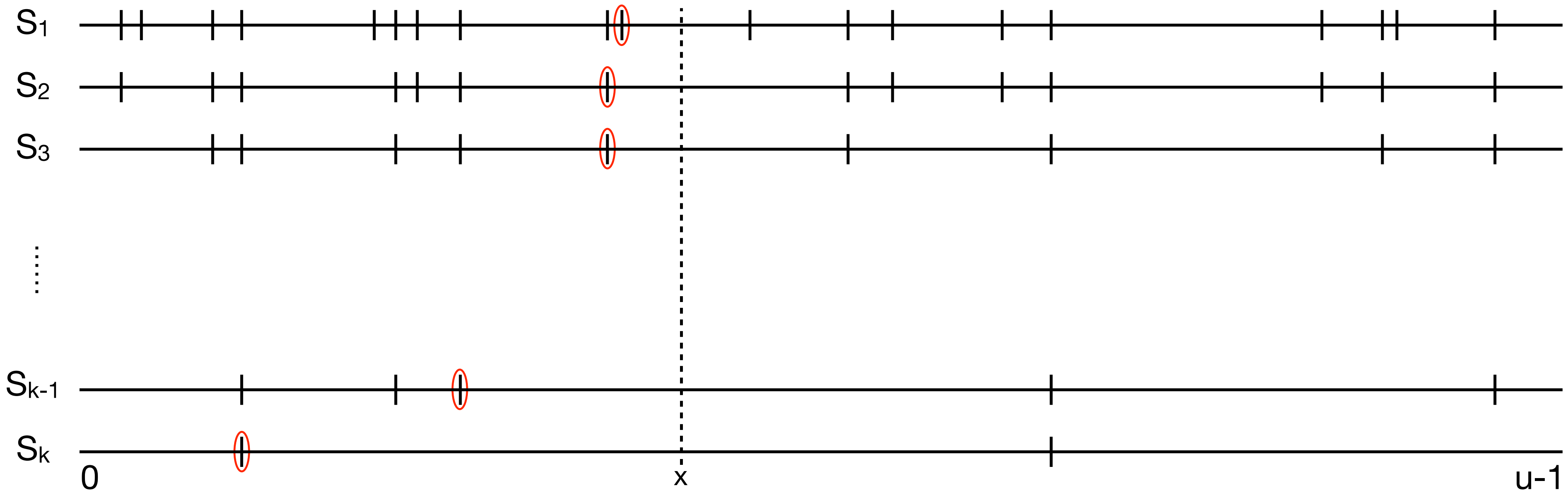

$|S_i| = n_i$ and $n_1 + n_2 + \cdots + n_k = n$

# Predecessor in Nested Sets

- Goal. Predecessor in nested sets with O(n) space and O(log n + k) query time.
- Solution in 3 steps.
  - Sorted arrays. Slow and linear space.
  - Tabulation. Fast but too much space.
  - Sorted arrays with bridges. Fast and little space.

# Solution 1: Sorted Arrays

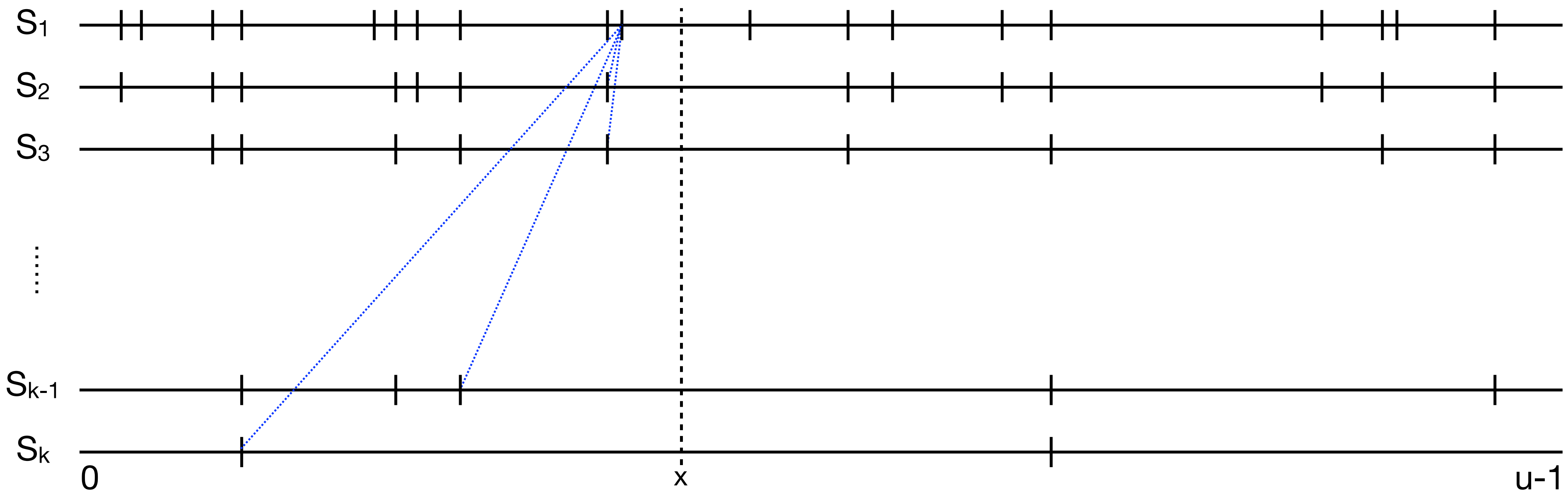
- Data structure. Sorted arrays for each set.
- Predecessor(x): Binary search in each array.
- Time. $O(\log n_1 + \log n_2 + \cdots + \log n_k) = O(k \log n)$
- Space. $O(n)$




$|S_i| = n_i$ and $n_1 + n_2 + \cdots + n_k = n$

# Solution 2: Tabulation

- Data structure. Sorted array on $S_1$ + each entry stores k-1 predecessors in $S_2, \ldots, S_k$.
- Predecessor(x): Binary search in $S_1$ array + report predecessors.
- Time. $O(\log n_1 + k) = O(\log n + k)$
- Space. $O(nk)$

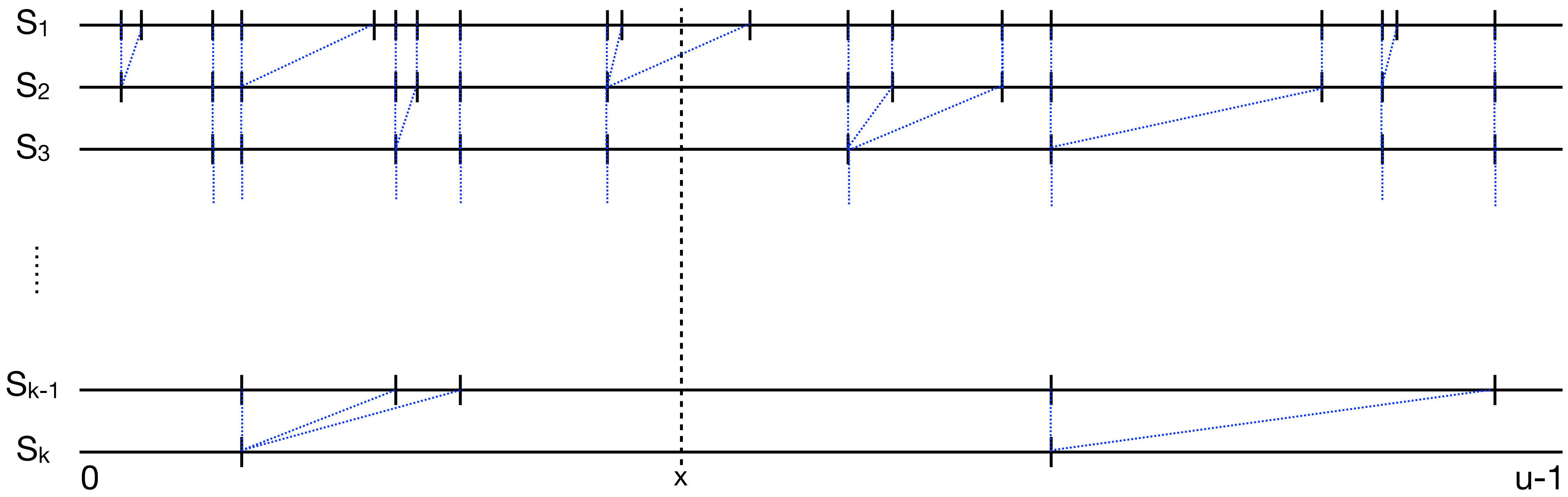



$|S_i| = n_i$ and $n_1 + n_2 + \cdots + n_k = n$

- Challenge. Can we get the best of both worlds?

# Solution 3: Sorted Arrays with Bridges

- Data structure. Sorted arrays for each set + bridges.
- Predecessor(x): Binary search in $S_1$ array + traverse bridges and report elements.
- Time. $O(\log n_1 + k) = O(\log n + k)$
- Space. $O(n)$




$|S_i| = n_i$ and $n_1 + n_2 + \cdots + n_k = n$

# Predecessor in Nested Sets

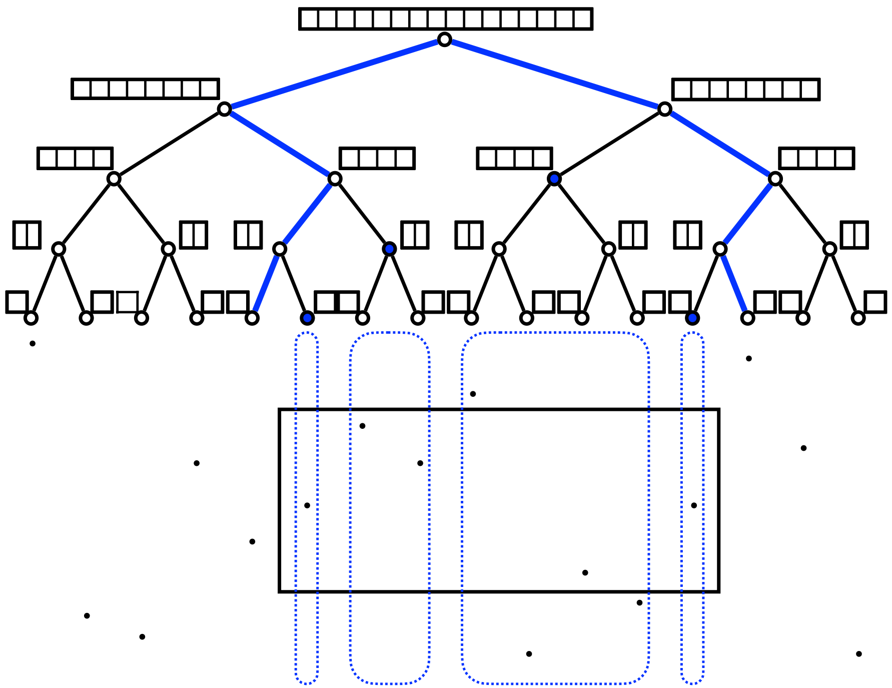
- Theorem. We can solve the predecessor in nested sets problem in
  - O(n) space.
  - O(log n + k) query time.
  - O(n log n) preprocessing time.

- Extensions.
  - Predecessor $\Longrightarrow$ 1D range reporting.
  - More tricks $\Longrightarrow$ works for non-nested sets. Called fractional cascading.

- Challenge. How can we use predecessor in nested sets for 2D range reporting?

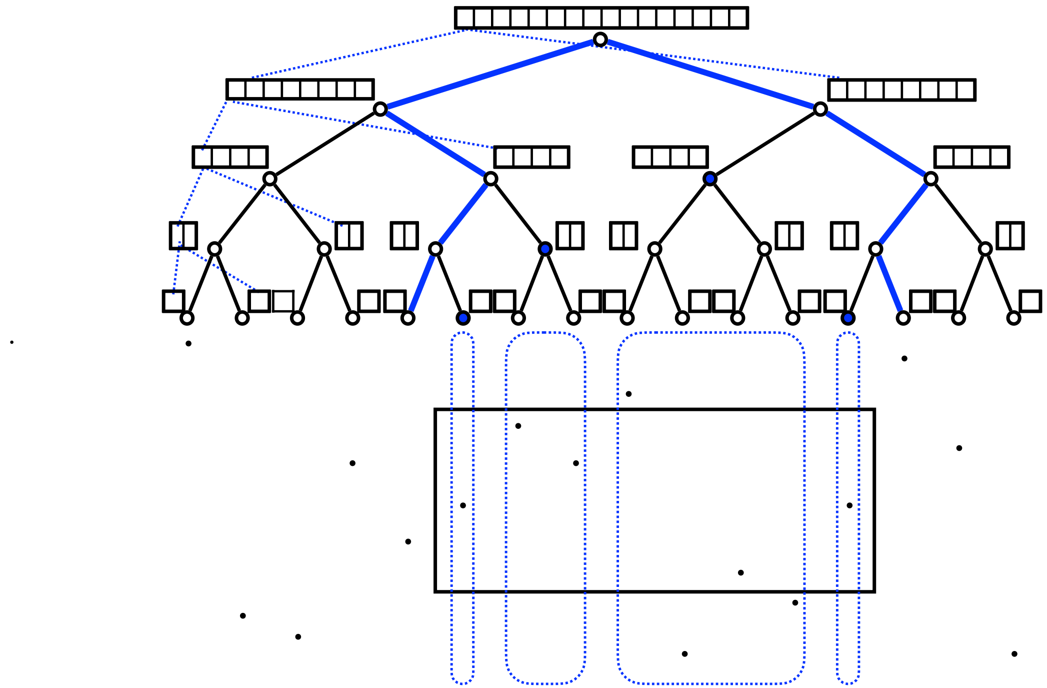
# 2D Range Reporting

- Goal. 2D range reporting in O(n log n) space and O(log n) time
- Idea. Consider node v with children $v_l$ and $v_r$.
  - Arrays at $v_l$ and $v_r$ are subsets of array at v.
  - All searches in arrays during a query are on the same y-range.



# 2D Range Reporting

- Data structure. 2D range tree with bridges.
  - Each point in array at v stores bridges to arrays in $v_l$ and $v_r$.
- Report($x_1$, $y_1$, $x_2$, $y_2$): As 2D range tree query
  - Binary search in root array + traverse bridges for remaining 1D queries.
- Time. O(log n + occ)
- Space. O(nlog n)
- Preprocessing. O(nlog n)



# 2D Range Reporting

- Theorem. We can solve the 2D range reporting problem in
  - O(n log n) space
  - O(log n + occ) time for queries.
  - O(n log n) preprocessing time.

- What can we do with only linear space?

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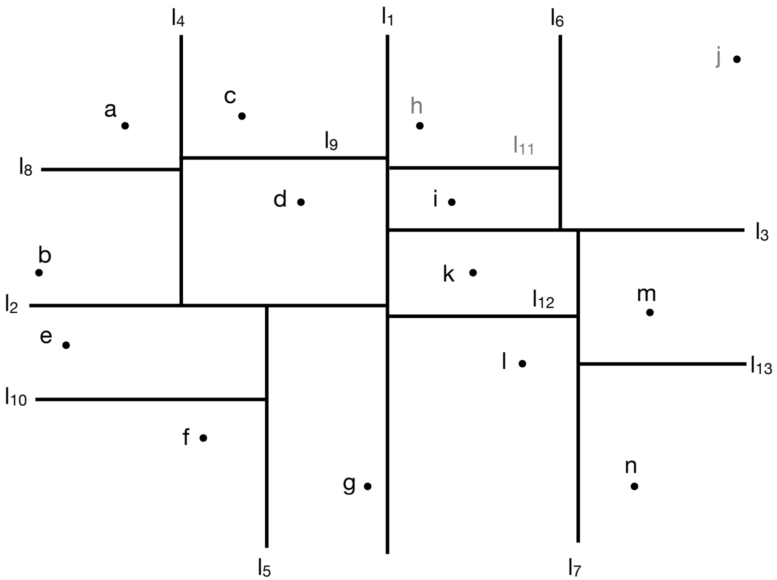
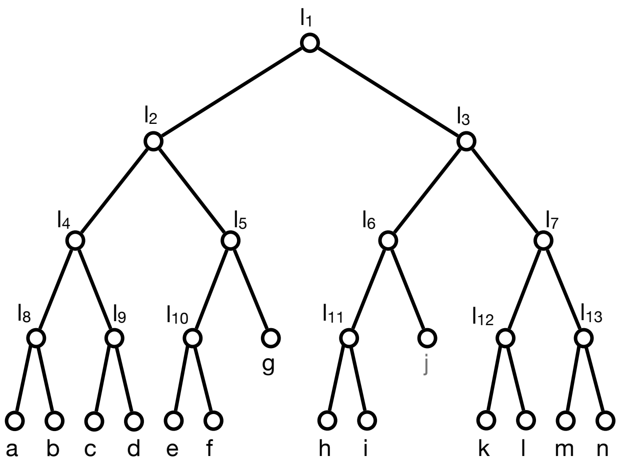
# kD Trees

- The 2D tree (k = 2).
  - A balanced binary tree over point set P.
  - Recursively partition P into rectangular regions containing (roughly) same number of points. Partition by alternating horizontal and vertical lines.
  - Each node in tree stores region and line.

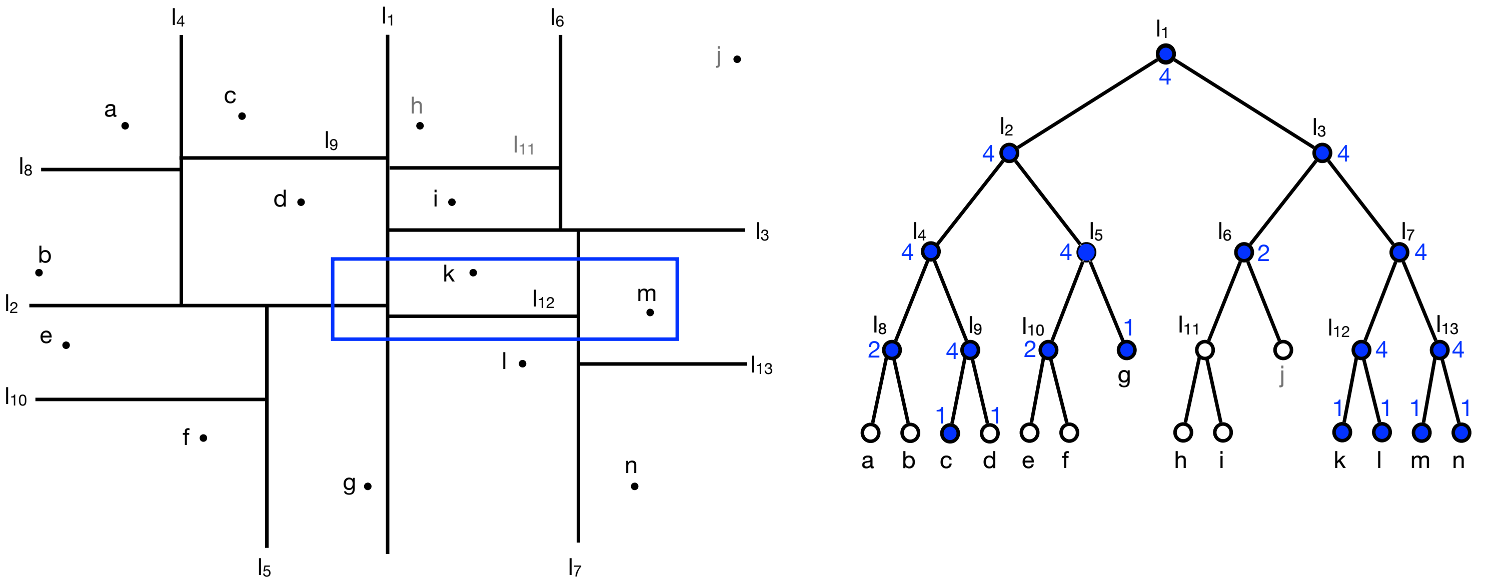



- Space. O(n)
- Preprocessing. O(n log n)

# kD Trees

- Report($x_1$, $y_1$, $x_2$, $y_2$): Traverse 2D tree starting at the root. At node v:
  - Case 1. v is a leaf: report the unique point in region(v) if contained in range.
  - Case 2. region(v) is disjoint from range: stop.
  - Case 3. region(v) is contained in range: report all points in region(v).
  - Case 4. region(v) intersects range, and v is not a leaf. Recurse left and right.




- Time. $O(\sqrt{n})$

# kD trees

- Theorem. We can solve the 2D range reporting problem in
  - O(n) space
  - $O(\sqrt{n} + \text{occ})$ time
  - O(n log n) preprocessing

# 2D Range Reporting

- Theorem. We can solve 2D range reporting in either
  - O(n log n) space and O(log n + occ) query time
  - O(n) space and $O(\sqrt{n} + \mathrm{occ})$ query time.
- Extensions.
  - More dimensions.
  - Inserting and deleting points.
  - Using word RAM techniques.
  - Other shapes (circles, triangles, etc.)

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