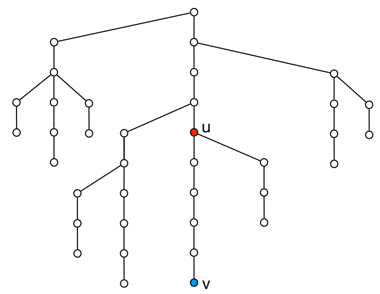
Level Ancestor

Philip Bille/Inge Li Gørtz

Level Ancestor

- Level ancestor problem. Preprocess rooted tree T with n nodes to support
  - LA(v,k): return the kth ancestor of node v.



LA(v,5) = u

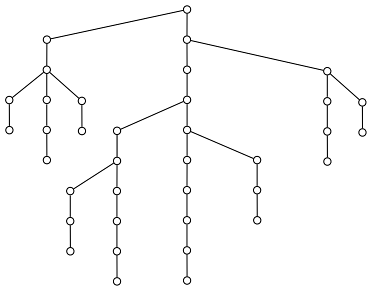
Level Ancestor

- Applications.
  - Basic primitive for navigating trees (any hierarchical data).
  - Illustration of wealth of techniques for trees.
    - Path decompositions.
    - Tree decomposition.
    - Tree encoding and tabulation.

Level Ancestor

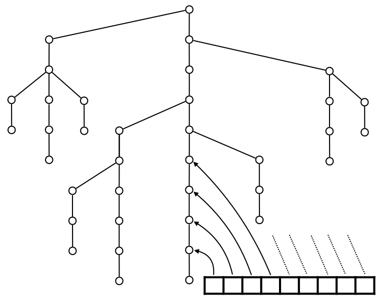
- Goal. Linear space and constant time.
- Solution in 7 steps (!).
  - No data structure. Very slow, litte space
  - Direct shortcuts. Very fast, lot of space.
  - ….
  - Ladder decomposition + jump pointers + top-bottom decomposition. Very fast, little space.

Solution 1: No Data Structure



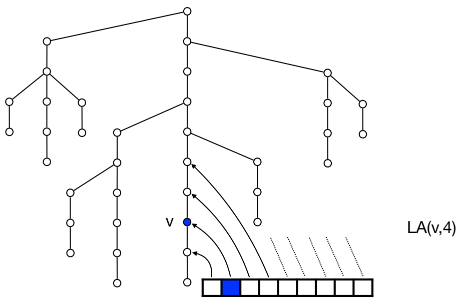
- Data structure. Store tree T (using pointers).
- LA(v,k): Walk up.
- Time. O(n)
- Space. O(n)

Solution 2: Direct Shortcuts



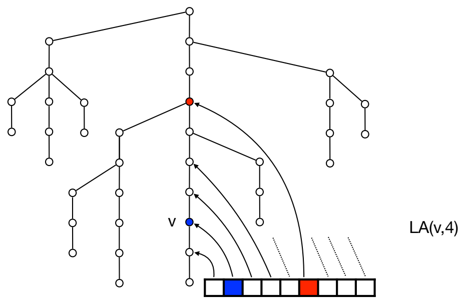
- Data structure. Store each root-to-leaf in array.
- LA(v,k): Jump up.
- Time. O(1)
- Space. O(n^2)

Solution 2: Direct Shortcuts



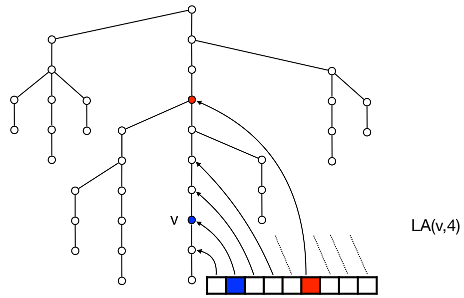
- Data structure. Store each root-to-leaf in array.
- LA(v,k): Jump up.
- Time. O(1)
- Space. O(n^2)

Solution 2: Direct Shortcuts



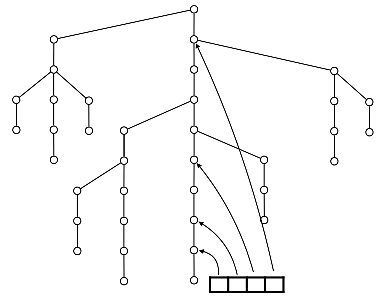
- Data structure. Store each root-to-leaf in array.
- LA(v,k): Jump up.
- Time. O(1)
- Space. O(n^2)

Solution 2: Direct Shortcuts



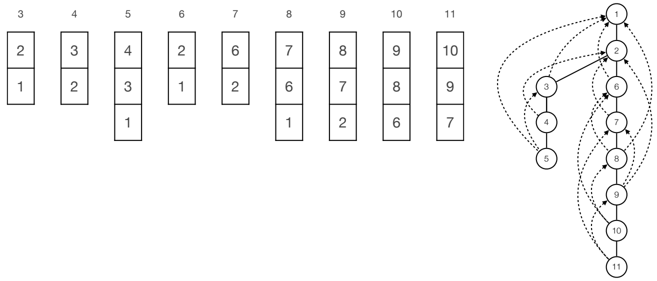
- Data structure. Store each root-to-leaf in array.
- LA(v,k): Jump up.
- Time. O(1)
- Space. $O(n^2)$

Solution 3: Jump Pointers



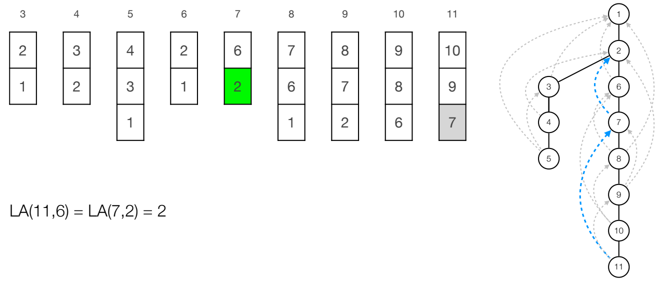
- Data structure. For each node v, store pointers to ancestors at distance 1,2,4, ..
- LA(v,k): Jump to most distant ancestor no further away than k. Repeat.
- Time. O(log n)
- Space. O(n log n)

Solution 3: Jump Pointers



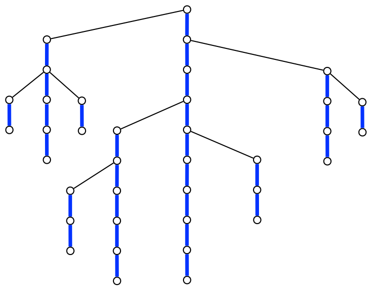
- Data structure. For each node v, store pointers to ancestors at distance 1,2,4, ..
- LA(v,k): Jump to most distant ancestor no further away than k. Repeat.
- Time. O(log n)
- Space. O(n log n)

Solution 3: Jump Pointers



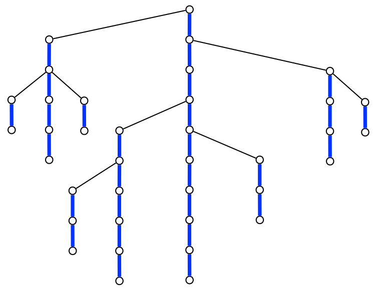
- Data structure. For each node v, store pointers to ancestors at distance 1,2,4, ..
- LA(v,k): Jump to most distant ancestor no further away than k. Repeat.
- Time. O(log n)
- Space. O(n log n)

Solution 4: Long Path Decomposition



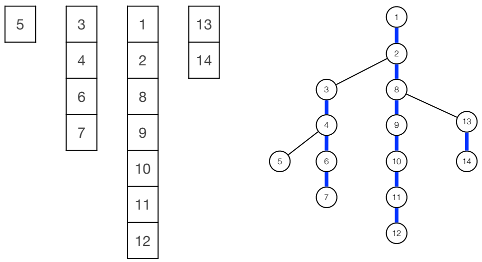
- Long path decomposition.
  - Find root-to-leaf path p of maximum length.
  - Recursively apply to subtrees hanging of p.
- Lemma. Any root-to-leaf path passes through at most $O(n^{1/2})$ long paths.
- Longest paths partition T $\Longrightarrow$ total length (number of nodes) of all longest paths is = n

Solution 4: Long Path Decomposition



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

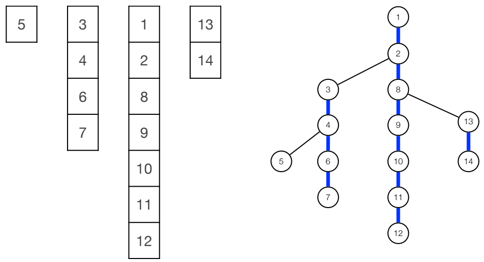
Solution 4: Long Path Decomposition



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

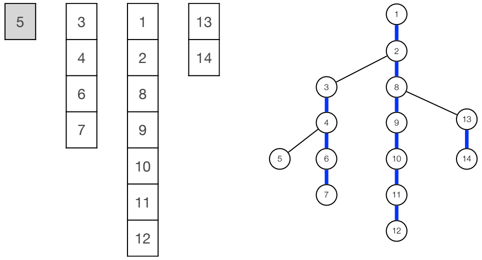
LA(5,4)



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

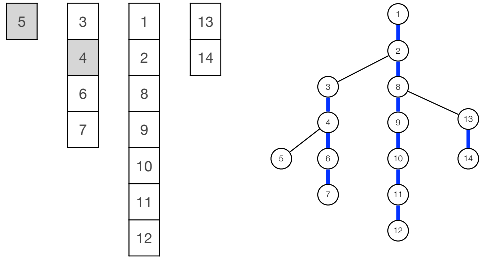
LA(5,4)



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

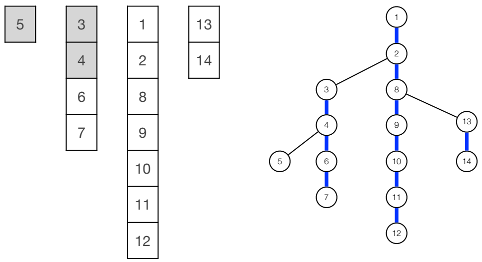
LA(5,4) = LA(4,3)



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

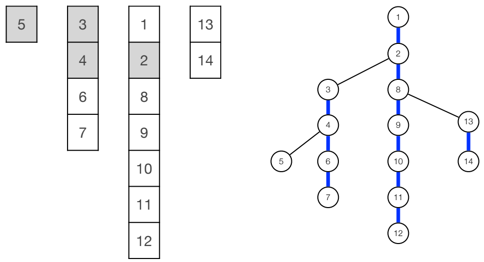
LA(5,4) = LA(4,3) = LA(3,2)



- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

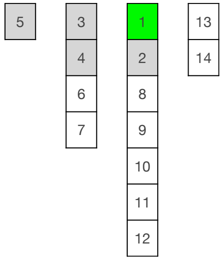
LA(5,4) = LA(4,3) = LA(3,2) = LA(2,1)

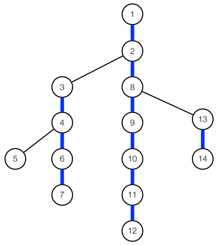


- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 4: Long Path Decomposition

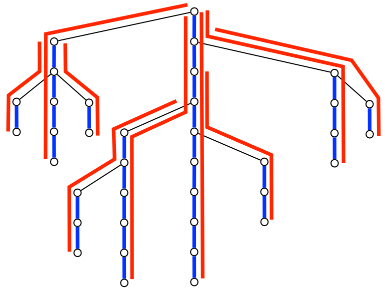
LA(5,4) = LA(4,4) = LA(3,3) = LA(2,1) = 1





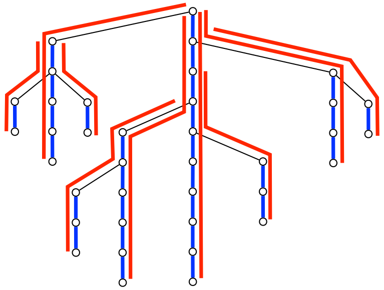
- Data structure. Store each long path in array.
- LA(v,k): Jump to kth ancestor or root of long path. Repeat.
- Time. $O(n^{1/2})$
- Space. O(n)

Solution 5: Ladder Decomposition



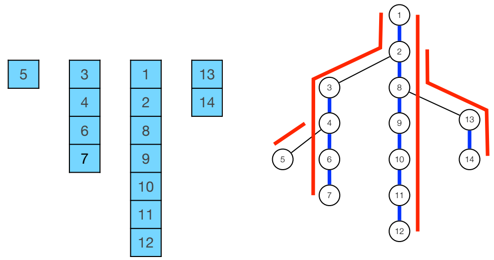
- Ladder decomposition.
  - Compute long path decomposition.
  - Double each long path.
- Lemma. Any root-to-leaf path passes through at most O(log n) ladders.
- Total length of ladders is ≤ 2n.

Solution 5: Ladder Decomposition



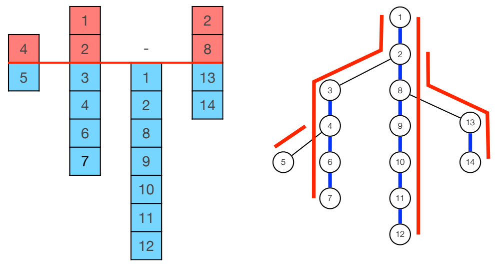
- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

Solution 5: Ladder Decomposition



- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

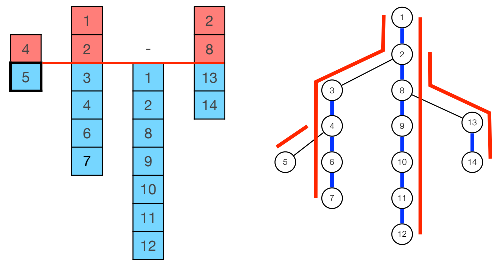
Solution 5: Ladder Decomposition



- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

Solution 5: Ladder Decomposition

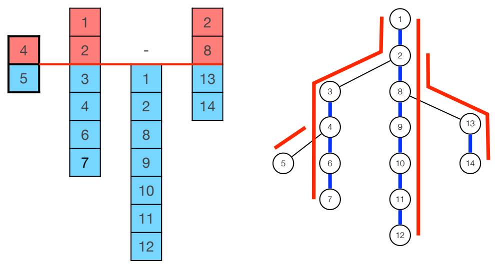
LA(5,4)



- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

Solution 5: Ladder Decomposition

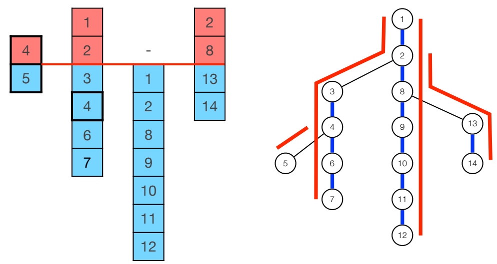
LA(5,4) = LA(4,3)



- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

Solution 5: Ladder Decomposition

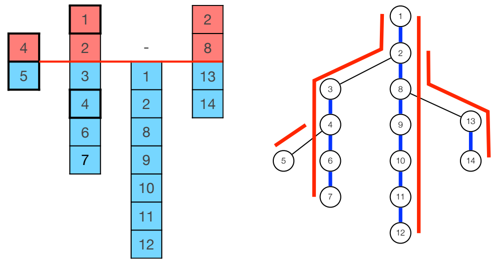
LA(5,4) = LA(4,3)



- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

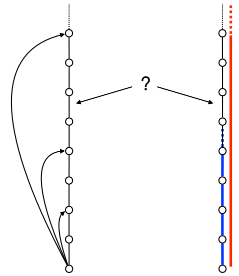
Solution 5: Ladder Decomposition

LA(5,4) = LA(4,3) = 1



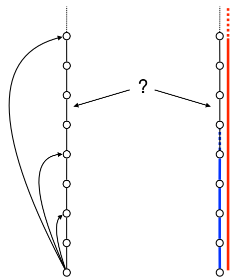
- Data structure.
  - Store each ladder in array.
  - Each node points to ladder corresponding to its longest path.
- LA(v,k): Jump to kth ancestor or root of ladder. Repeat.
- Time. O(log n)
- Space. O(n)

Solution 6: Ladder Decomposition + Jump Pointers



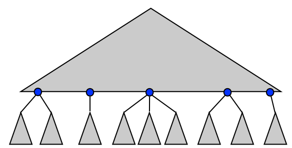
- Data structure. Ladder decomposition + Jump pointers.
- LA(v,k):
  - Jump to most distant ancestor not further away than k using jump pointer.
  - Jump to kth ancestor using ladder.
- Time. O(1)
- Space. O(n) + O(n log n) = O(n log n)

Solution 6: Ladder Decomposition + Jump Pointers



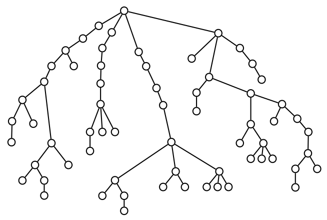
- Correctness.
  - A node at height x is on a ladder of height at least 2x.
  - After jump we are at a node of height at least k/2.
  - => after jump we are at a ladder that contains our goal.

Solution 7: Top-Bottom Decomposition



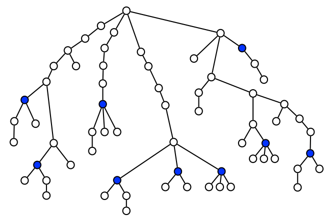
- Jump nodes. Maximal deep nodes with ≥ 1/4 log n descendants.
- Top tree. Jump nodes + ancestors.
- Bottom trees. Below top tree.

Solution 7: Top-Bottom Decomposition



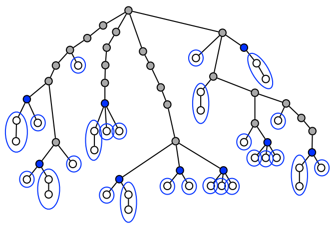
- Jump nodes. Maximal deep nodes with ≥ 1/4 log n descendants.
- Top tree. Jump nodes + ancestors.
- Bottom trees. Below top tree.

Solution 7: Top-Bottom Decomposition



- Jump nodes. Maximal deep nodes with ≥ 1/4 log n descendants.
- Top tree. Jump nodes + ancestors.
- Bottom trees. Below top tree.

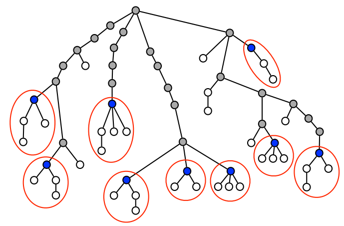
Solution 7: Top-Bottom Decomposition



- Jump nodes. Maximal deep nodes with ≥ 1/4 log n descendants.
- Top tree. Jump nodes + ancestors.
- Bottom trees. Below top tree.

- Size of each bottom tree < 1/4 log n.

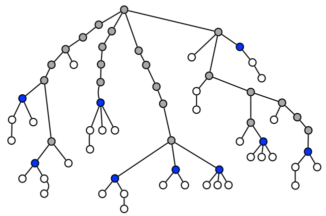
Solution 7: Top-Bottom Decomposition



- Jump nodes. Maximal deep nodes with ≥ 1/4 log n descendants.
- Top tree. Jump nodes + ancestors.
- Bottom trees. Below top tree.

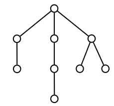
- Size of each bottom tree < 1/4 log n.
- Number of jump nodes is at most O(n/log n).

Solution 7: Top-Bottom Decomposition



- Data structure for top.
  - Ladder decomposition + Jump pointers for jump nodes.
  - For each internal node pointer to some jump node below.
- LA(v,k) in top:
  - Follow pointer to jump node below v.
  - Jump pointer + ladder solution.
- Time. O(1)
- Space. O(n) + (n/log n · log n) = O(n)

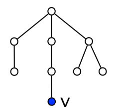
Solution 7: Top-Bottom Decomposition



(()) ((())) (()())

- Tree encoding. Encode each bottom tree B using balanced parentheses representation.
  - < 2 · 1/4 log n = 1/2 log n bits.

Solution 7: Top-Bottom Decomposition



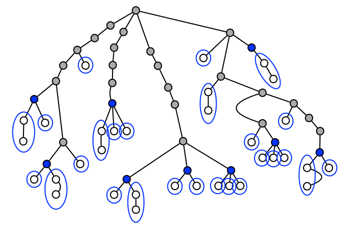
v has preorder number 5 = 0101_2

(()) ((())) (()())

`Code(B,v,2) = 0011000111001011 0101 0010`

- Tree encoding. Encode each bottom tree B using balanced parentheses representation.
  - < 2 · 1/4 log n = 1/2 log n bits.
- Integer encoding. Encode inputs v and k to LA
  - < 2 · log(1/4log n) < 2 loglog n bits.
- LA encoding. Concatenate into code(B, v, k)
  - ⟹ |code(B, v, k)| < 1/2 log n + 2 log log n bits.

Solution 7: Top-Bottom Decomposition



- Data structure for bottom.
  - Build table A s.t. A[code(B, v, k)] = LA(v, k) in bottom tree B.
- LA(v,k) in bottom: Lookup in A.
- Time. O(1)
- Space. $2^{|code|} < 2^{1/2 \log n + 2 \log \log n} = n^{1/2} \log^2 n = o(n)$.
- Combine bottom and top data structures ⟹ O(n) space and O(1) query time.

Solution 7: Top-Bottom Decomposition

- Theorem. We can solve the level ancestor problem in linear space and constant query time.