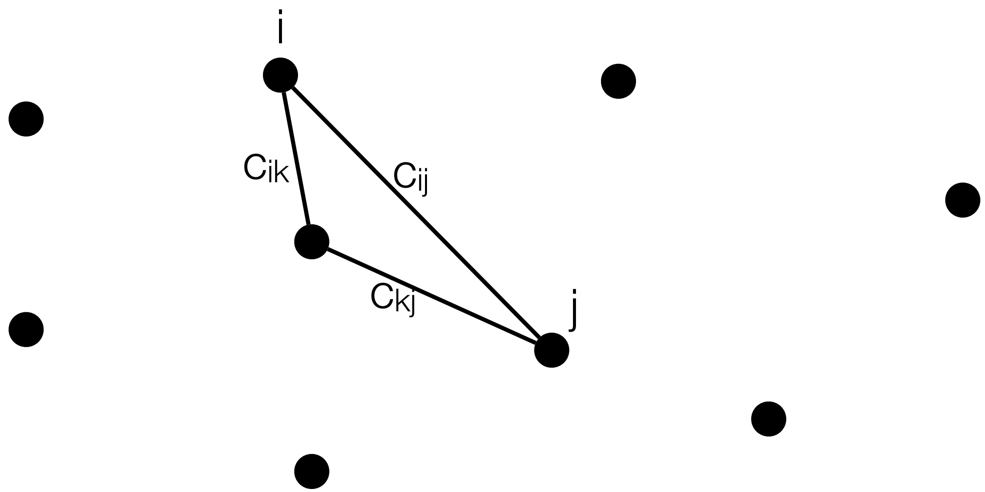
Traveling salesman problem

Inge Li Gørtz

Traveling Salesman Problem (TSP)



- Set of cities {1,…,n}
- $c_{ij} \geq 0$: cost of traveling from i to j.
- c_{ij} a metric:
  - $c_{ii} = 0$
  - $c_{ij} = c_{ji}$
  - $c_{ij} \leq c_{ik} + c_{kj}$ (triangle inequality)
- Goal: Find a *tour of minimum cost visiting every city exactly once*.

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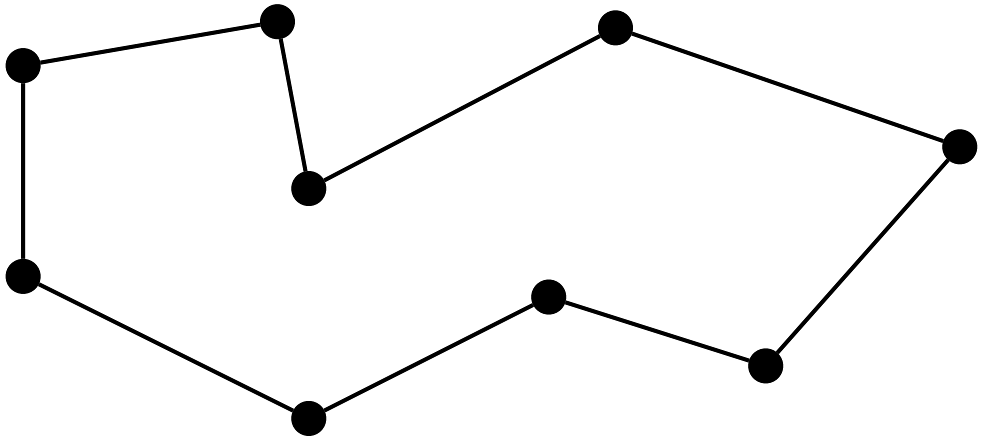

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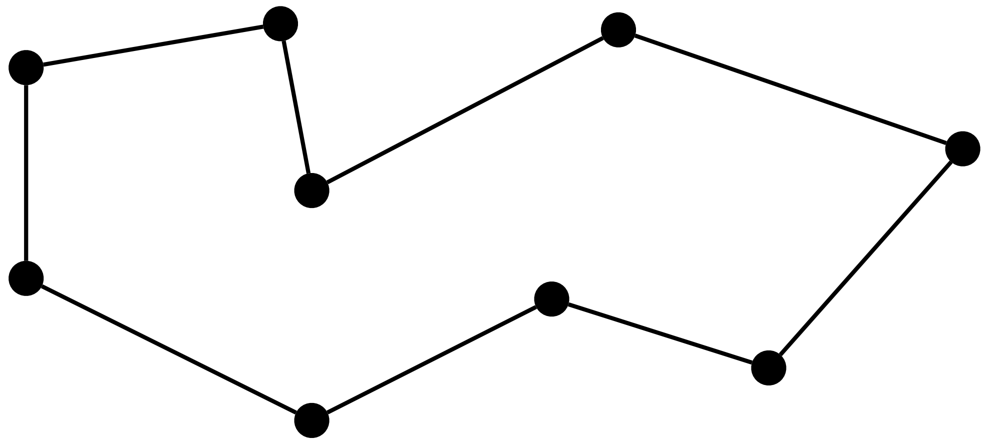
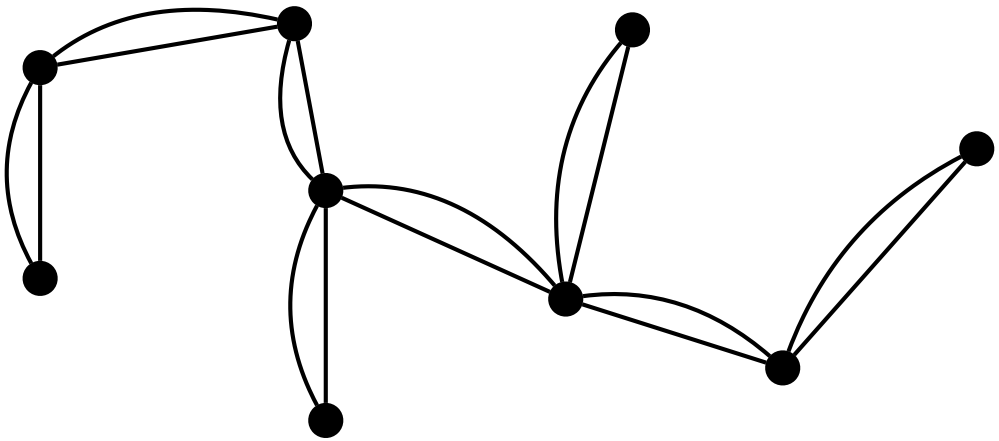
Double tree algorithm



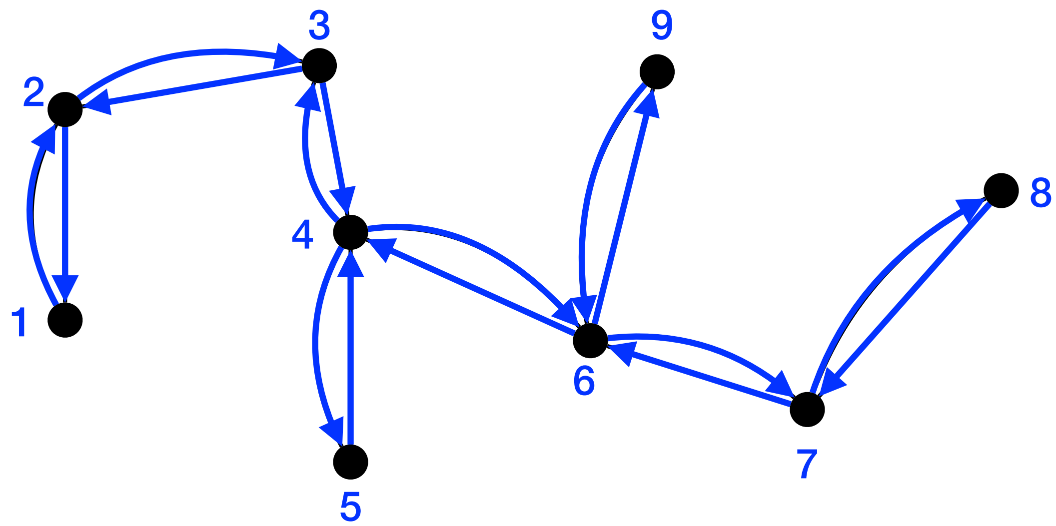
- MST is a lower bound on TSP.
    - Deleting an edge e from OPT gives a spanning tree.
    - OPT $\geq$ OPT - $c_e \geq$ MST.

Double tree algorithm



- Double tree algorithm
  - Compute MST T.
  - Double edges of T
  - Construct Euler tour **τ** (a tour visiting every edge exactly once).

Double tree algorithm



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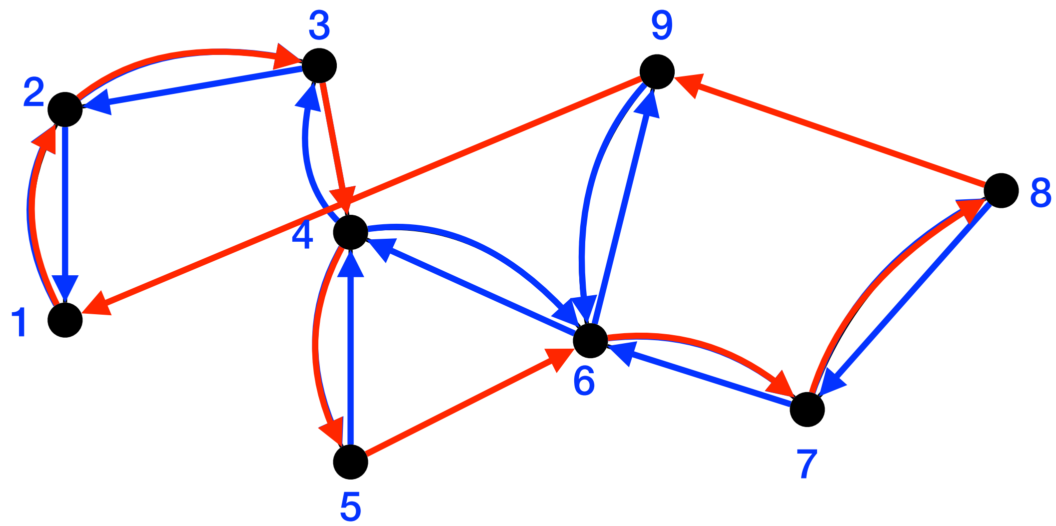
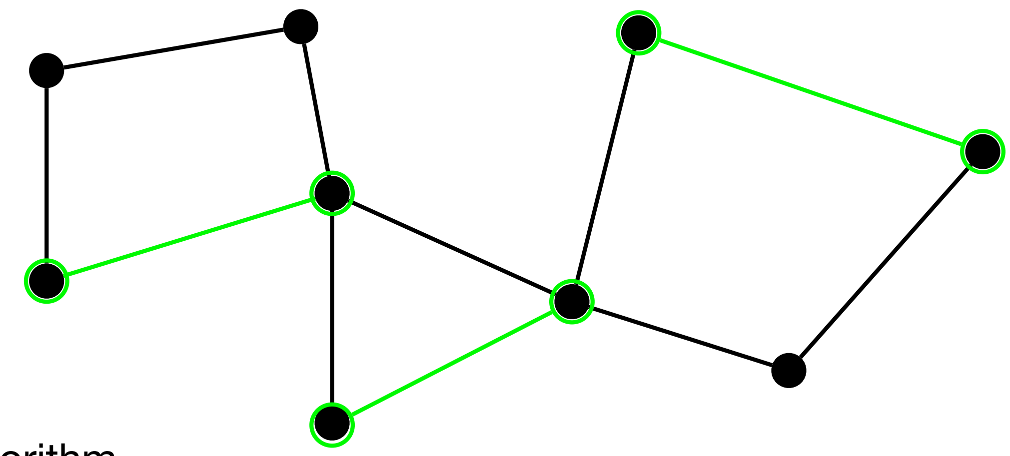


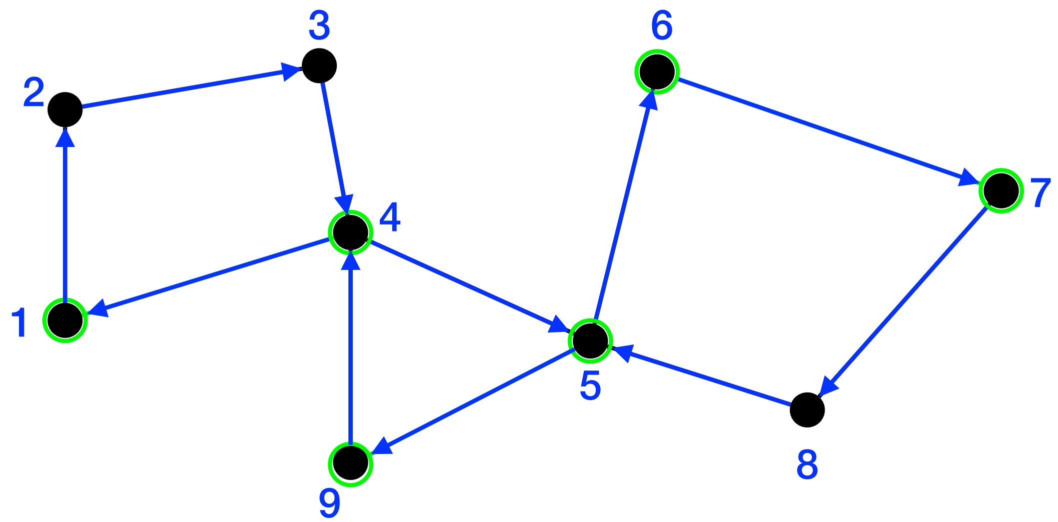
- Double tree algorithm
  - Compute MST T.
  - Double edges of T
  - Construct Euler tour **τ** (a tour visiting every edge exactly once).
  - Shortcut **τ** such that each vertex only visited once (**τ’**)
- length(**τ’**) ≤ length(**τ**) = 2 cost(T) ≤ 2 OPT.
- The double tree algorithm is a 2-approximation algorithm for TSP.

Christofides' algorithm



- Christofides' algorithm
  - Compute MST T.
  - No need to double all edges:
    - Enough to turn it into an Eulerian graph: *A graph Eulerian if there is a traversal of all edges visiting every edge exactly once.*
      - G Eulerian iff G connected and all nodes have even degree.
    - Consider set O of all odd degree vertices in T.
    - Find minimum cost perfect matching M on O.
      - Matching: no edges share an endpoint.
      - Perfect: all vertices matched.
      - Perfect matching on O exists: Number of odd vertices in a graph is even.
    - T + M is Eulerian (all vertices have even degree).

Christofides' algorithm



- Christofides' algorithm
  - Compute MST T.
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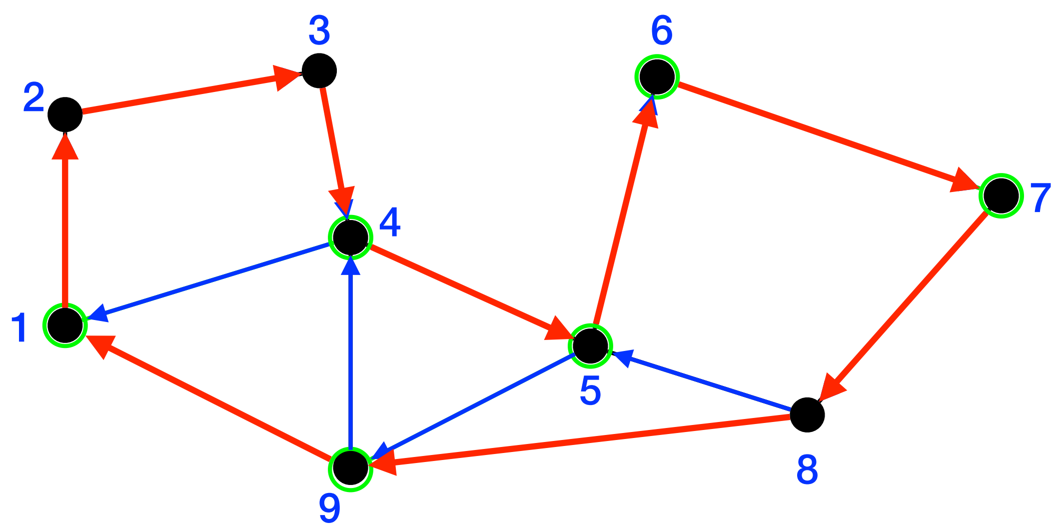
  - Shortcut such that each vertex only visited once (τ')

Christofides' algorithm



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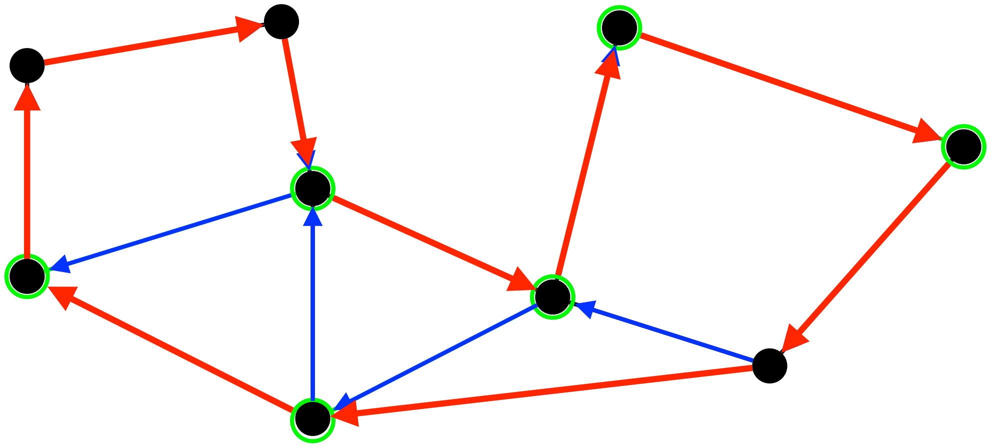
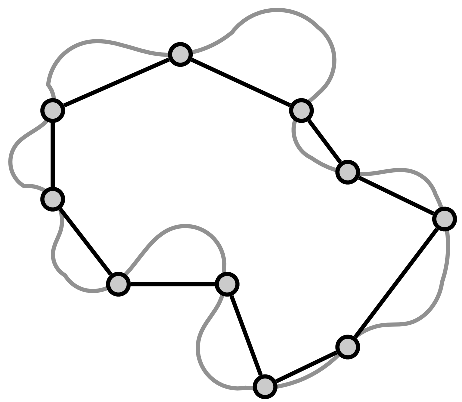


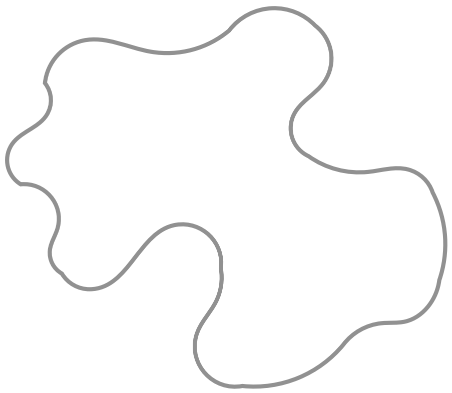
- Christofides' algorithm
  - Compute MST T.
  - O = {odd degree vertices in T}.
  - Compute minimum cost perfect matching **M** on O.
  - Construct Euler tour **τ**
  - Shortcut such that each vertex only visited once (**τ'**)
- length(**τ'**) ≤ length(**τ**) = cost(T) + cost(**M**) ≤ OPT + cost(**M**).

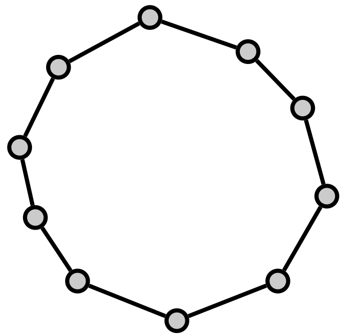
Analysis of Christofides' algorithm



- cost(M) ≤ OPT/2.
  - OPT_O = OPT restricted to O.
  - OPT_O ≤ OPT.

Analysis of Christofides' algorithm





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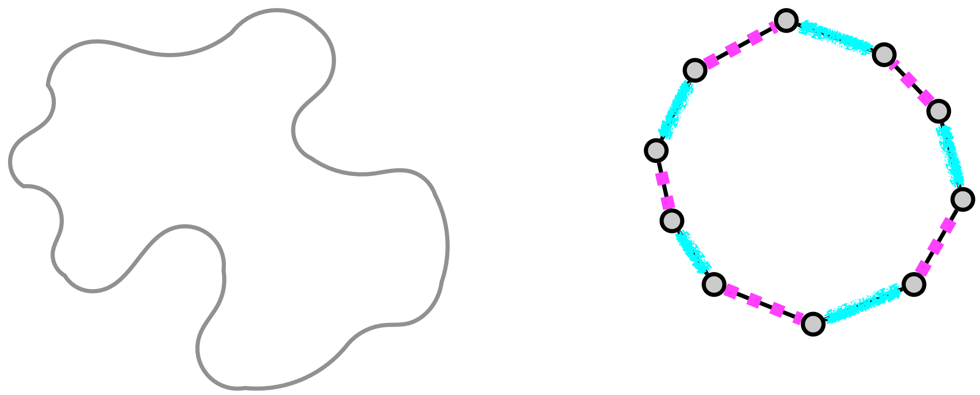
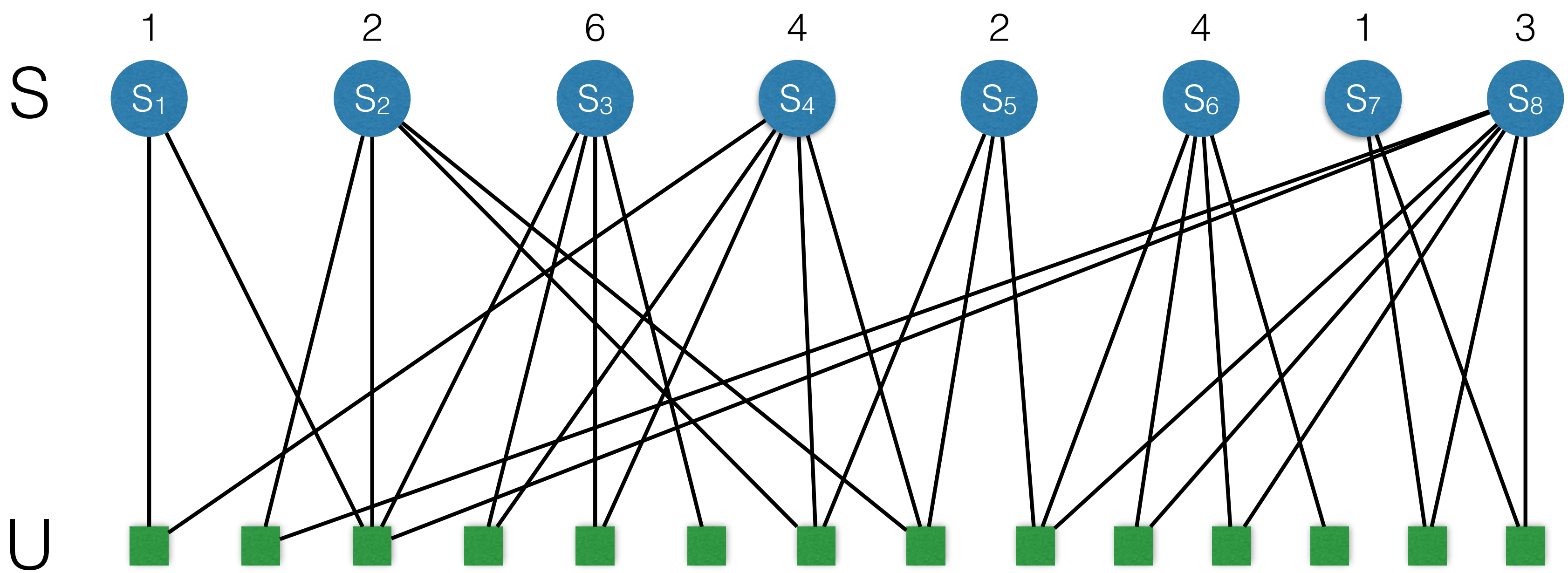


- cost(M) $\leq$ OPT/2:
  - OPT_O = OPT restricted to O.
  - $OPT_O \leq OPT$.
  - can partition OPT_O into two perfect matchings O_1 and O_2.
  - cost(M) $\leq$ min(cost(O_1), cost(O_2)) $\leq$ OPT/2.
- length(τ') $\leq$ length(τ) = cost(T) + cost(M) $\leq$ OPT + OPT/2 = 3/2 OPT.
- Christofides' algorithm is a 3/2-approximation algorithm for TSP.

Set cover

Set cover problem

- Set U of n elements.
- Subsets of U: $S_1,\ldots,S_m$.
- Each set S_i has a weight $w_i \geq 0$.
- Set cover. A collection of subsets C whose union is equal to U.
- Goal. find set cover of minimum weight.



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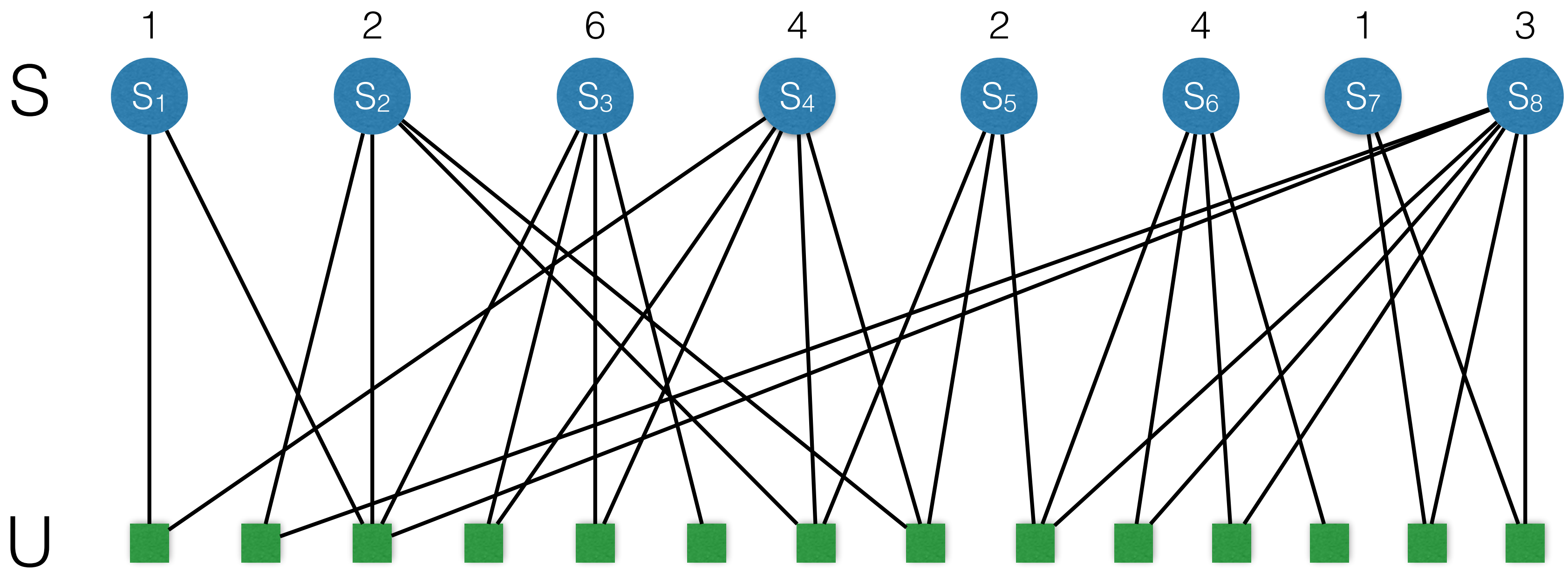

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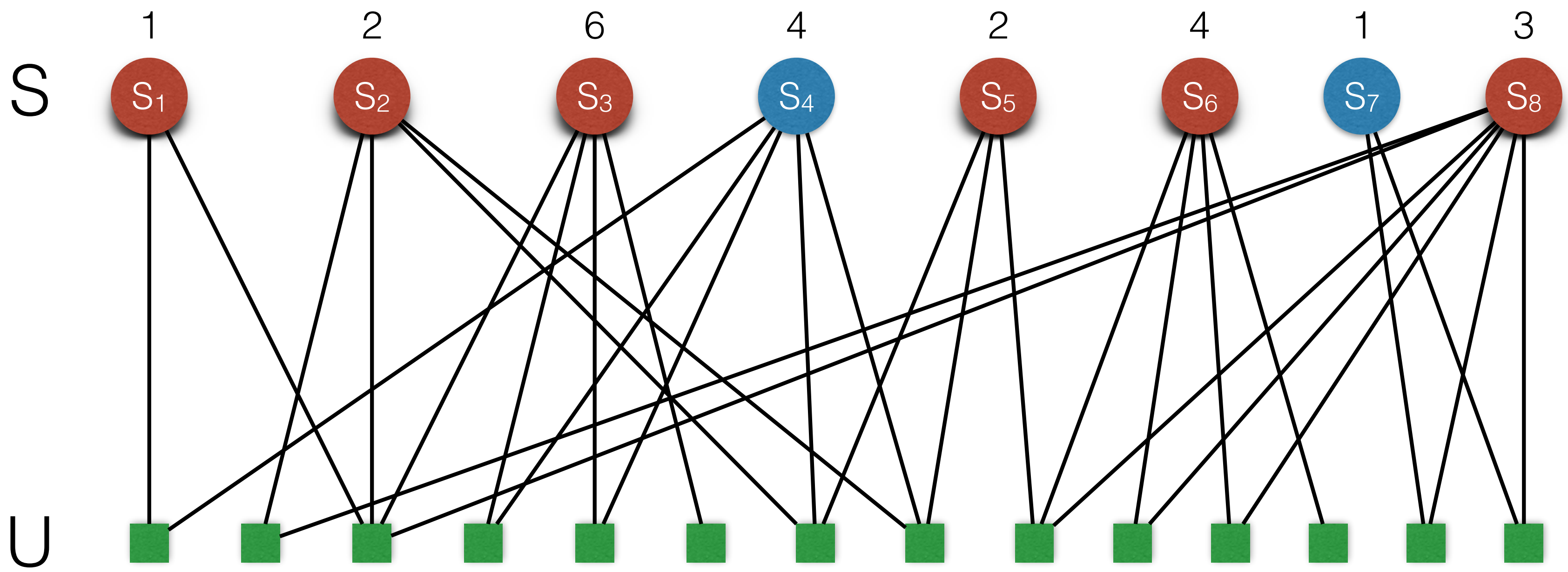

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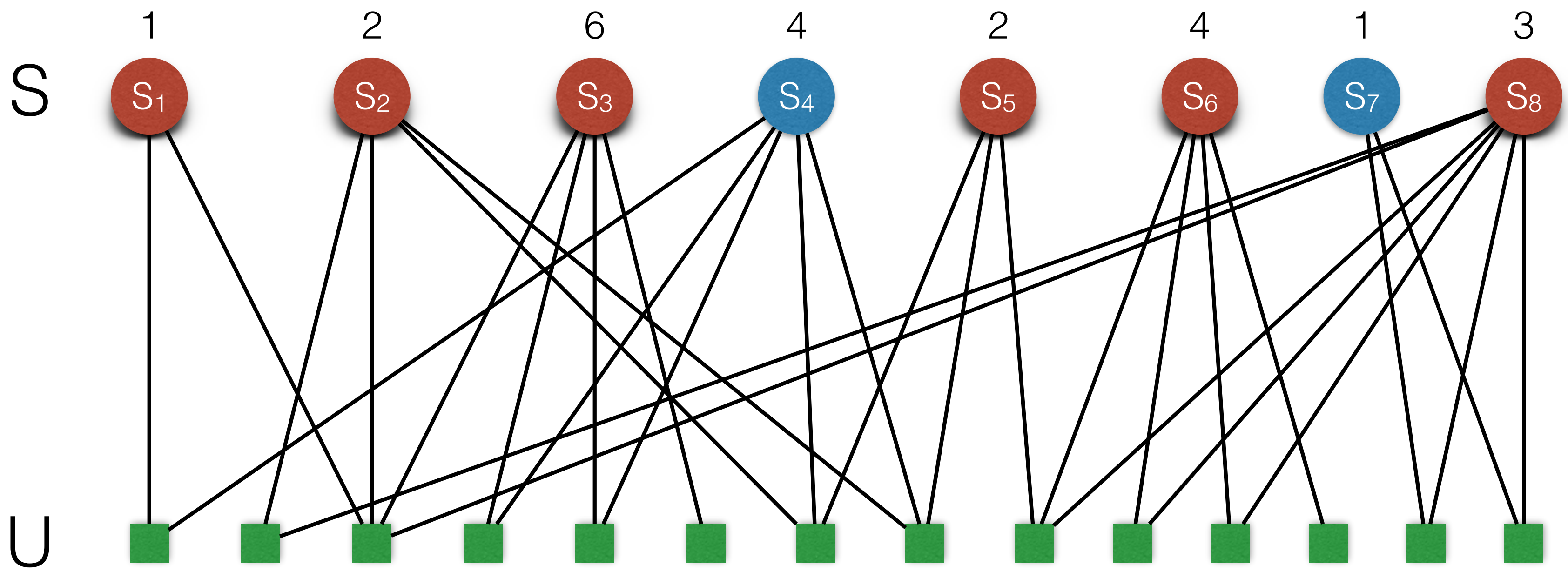

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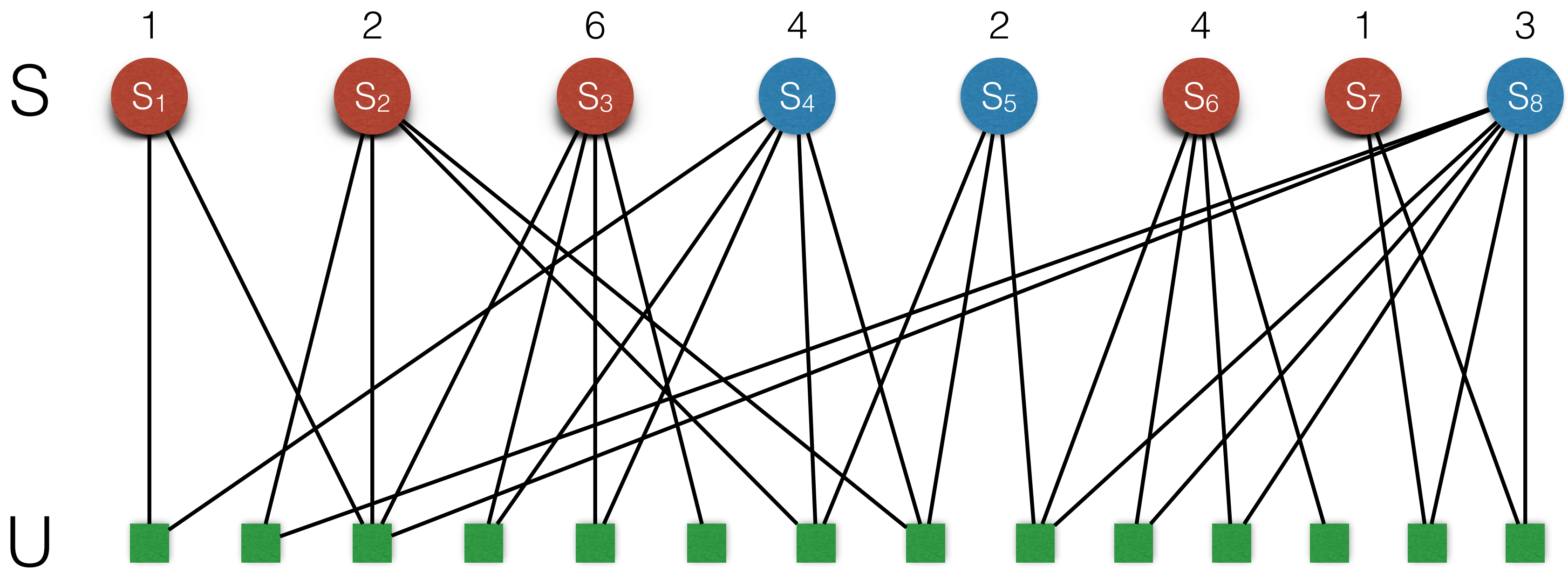

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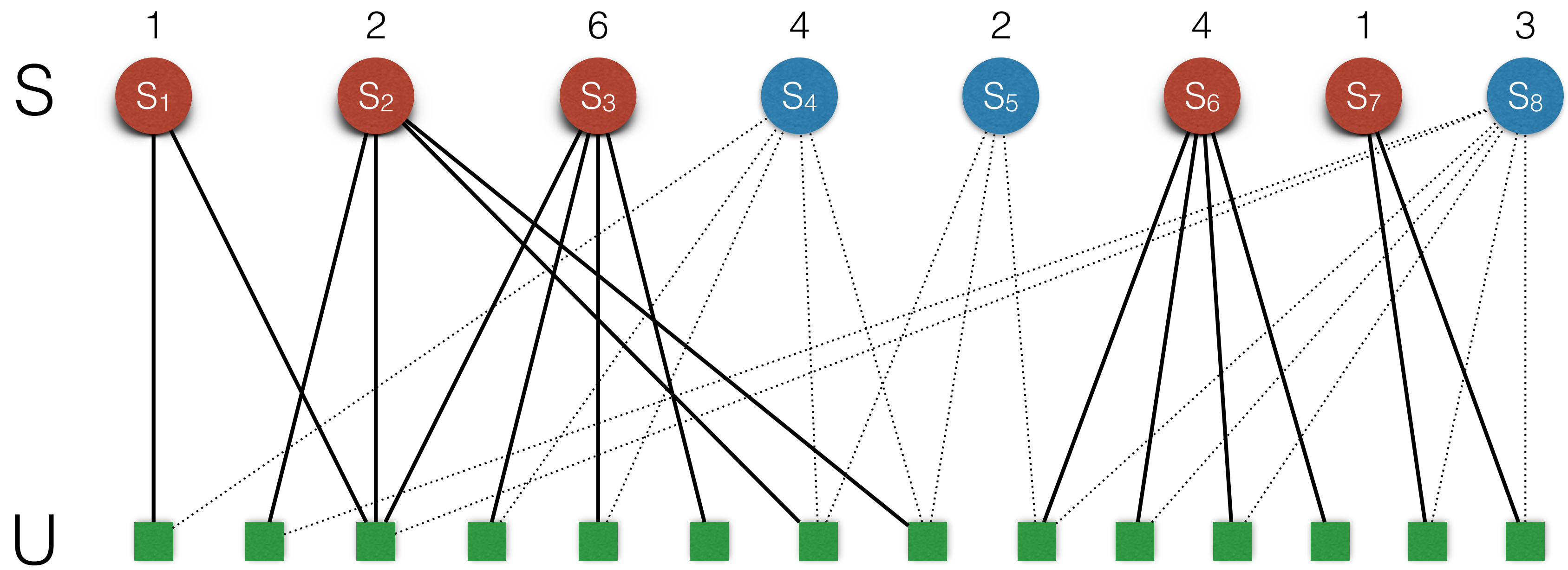
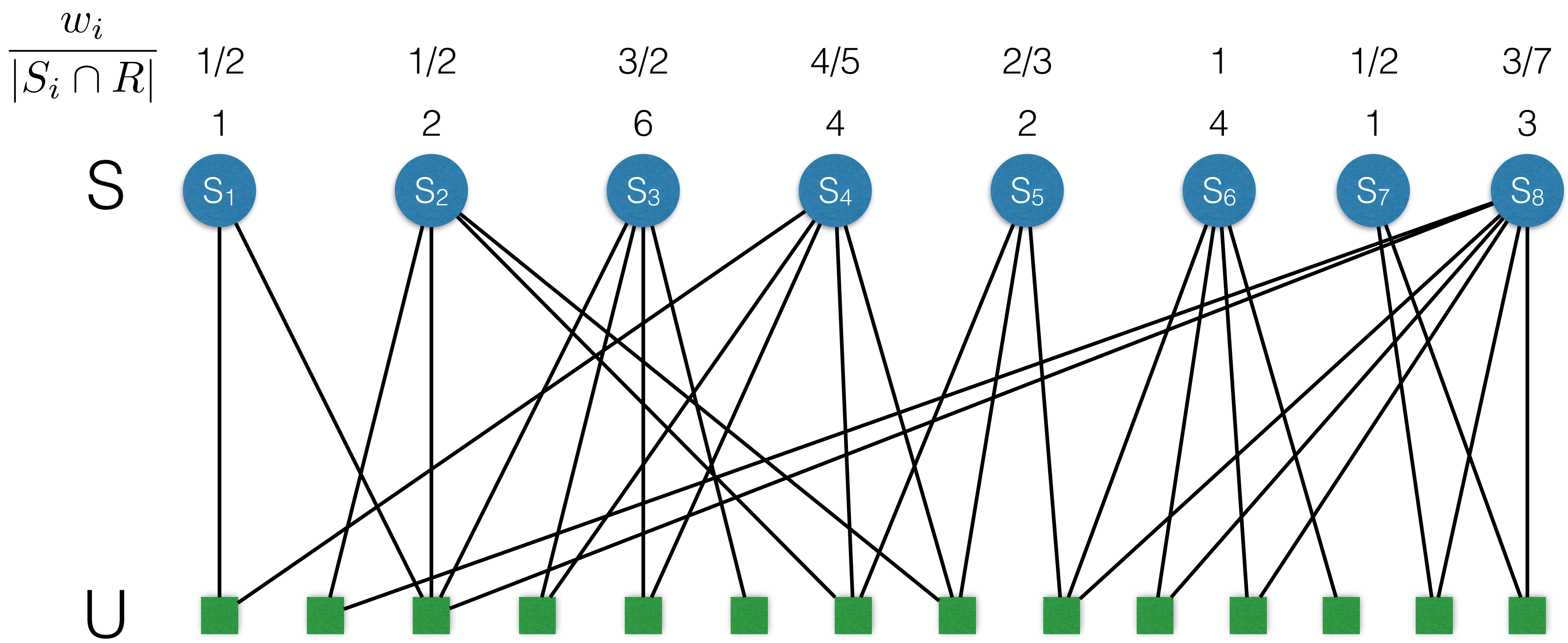


Set Cover: Greedy algorithm



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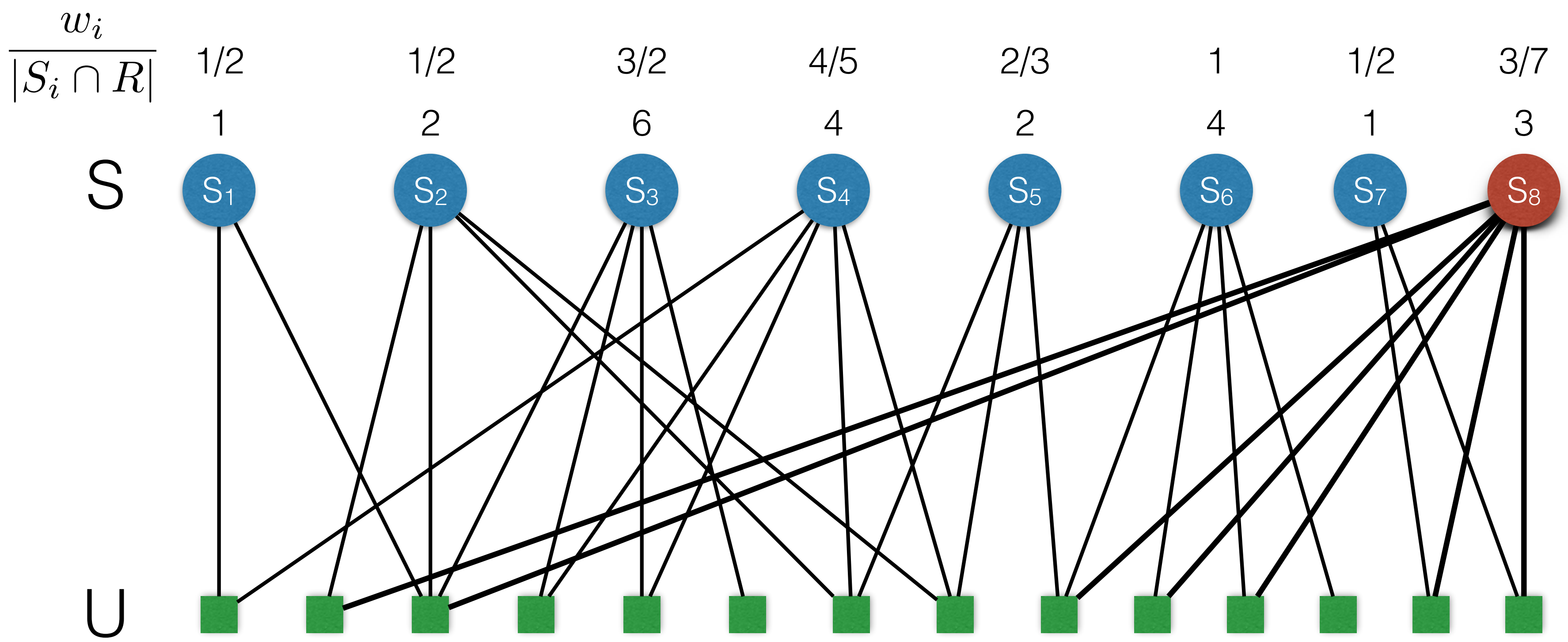

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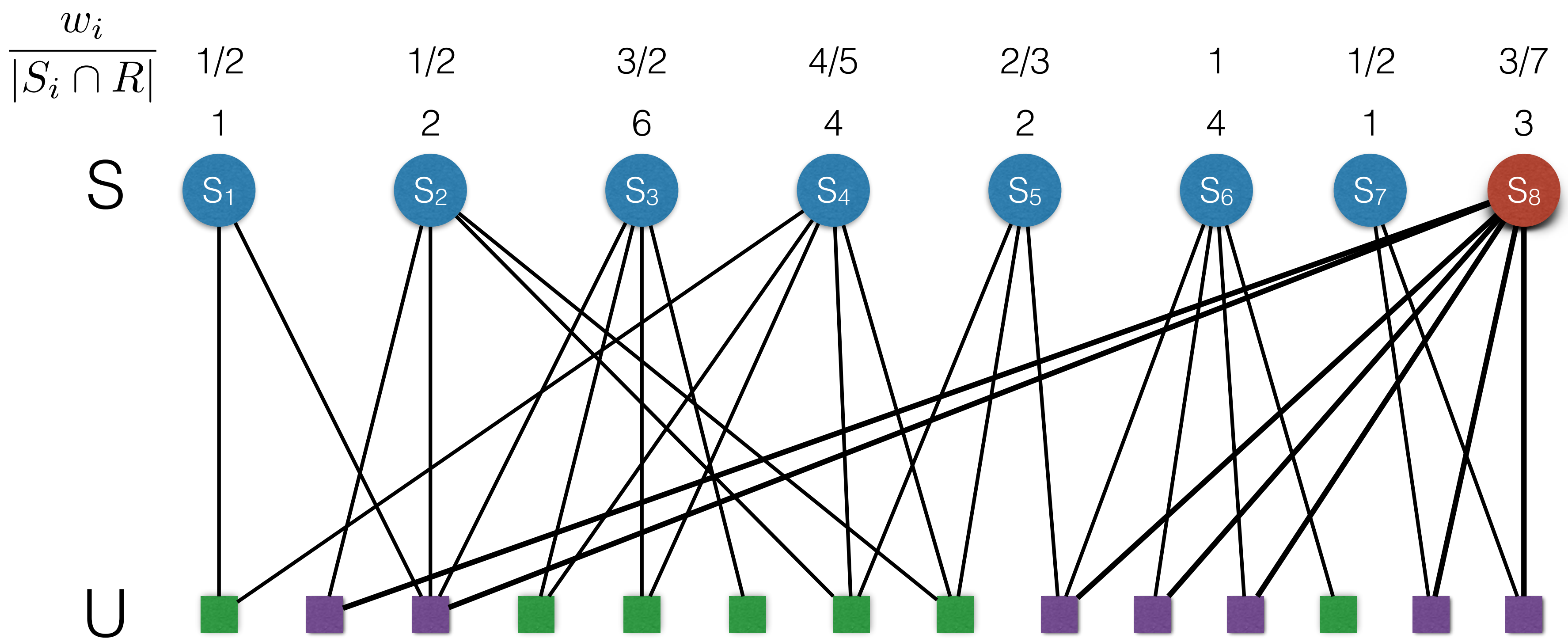

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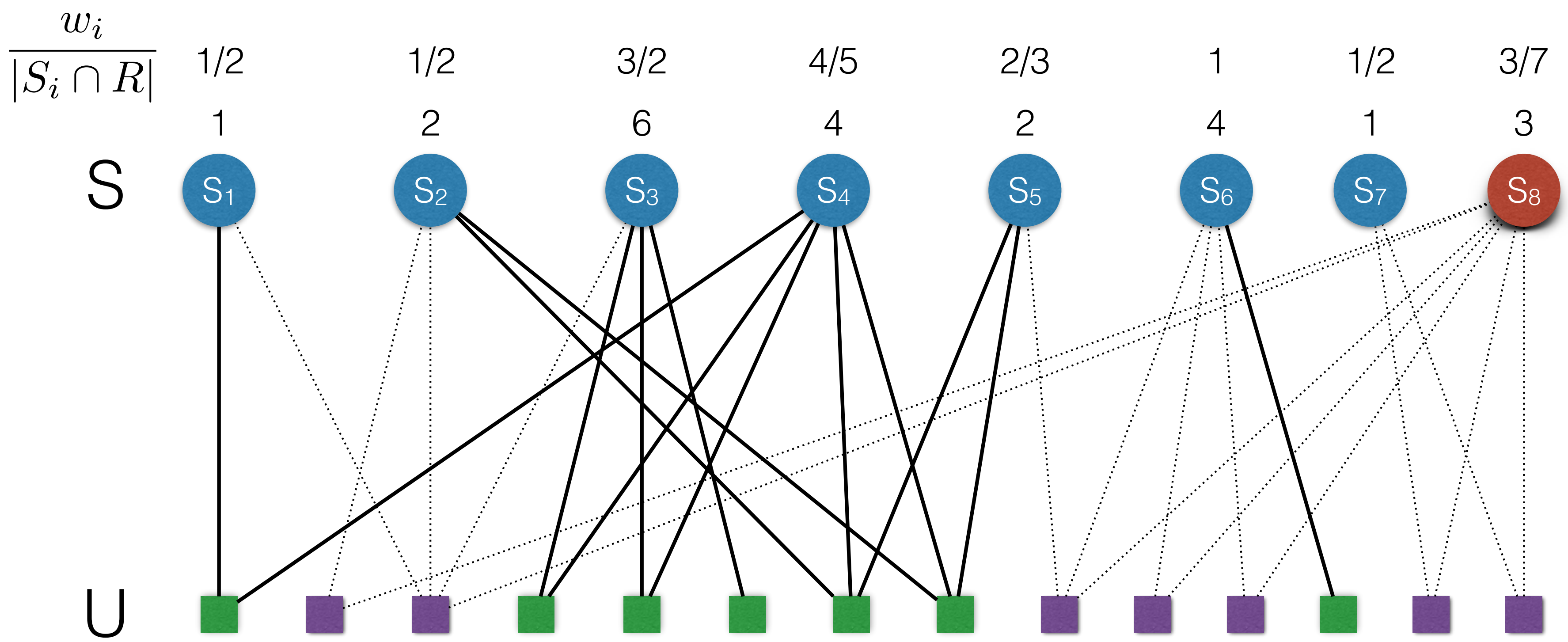

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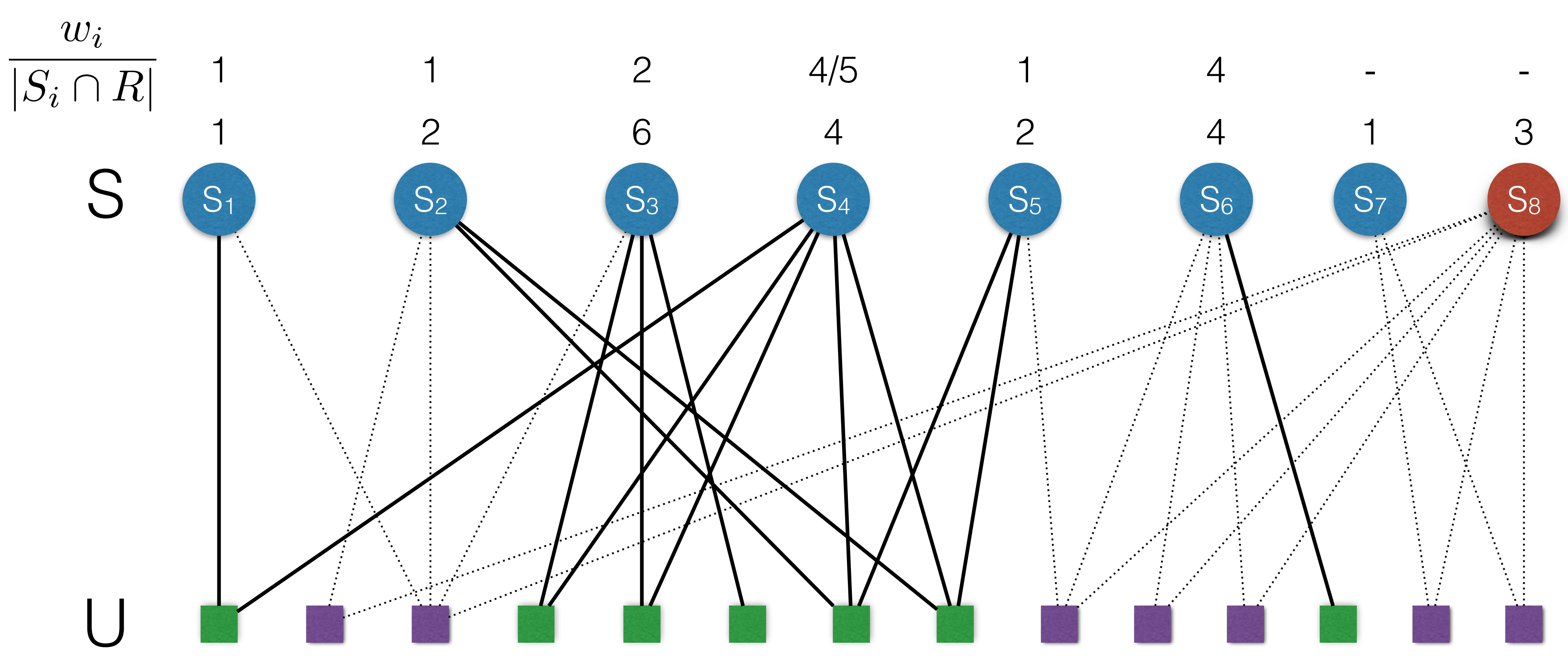

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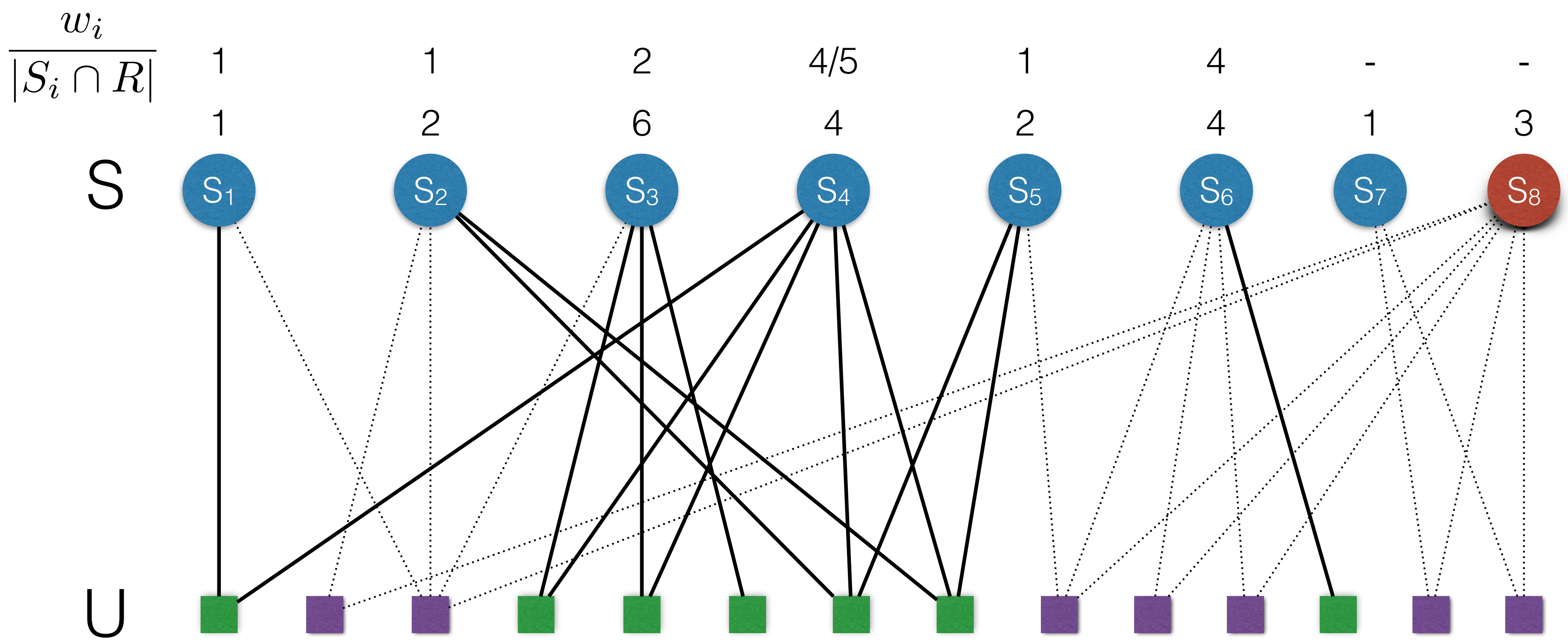

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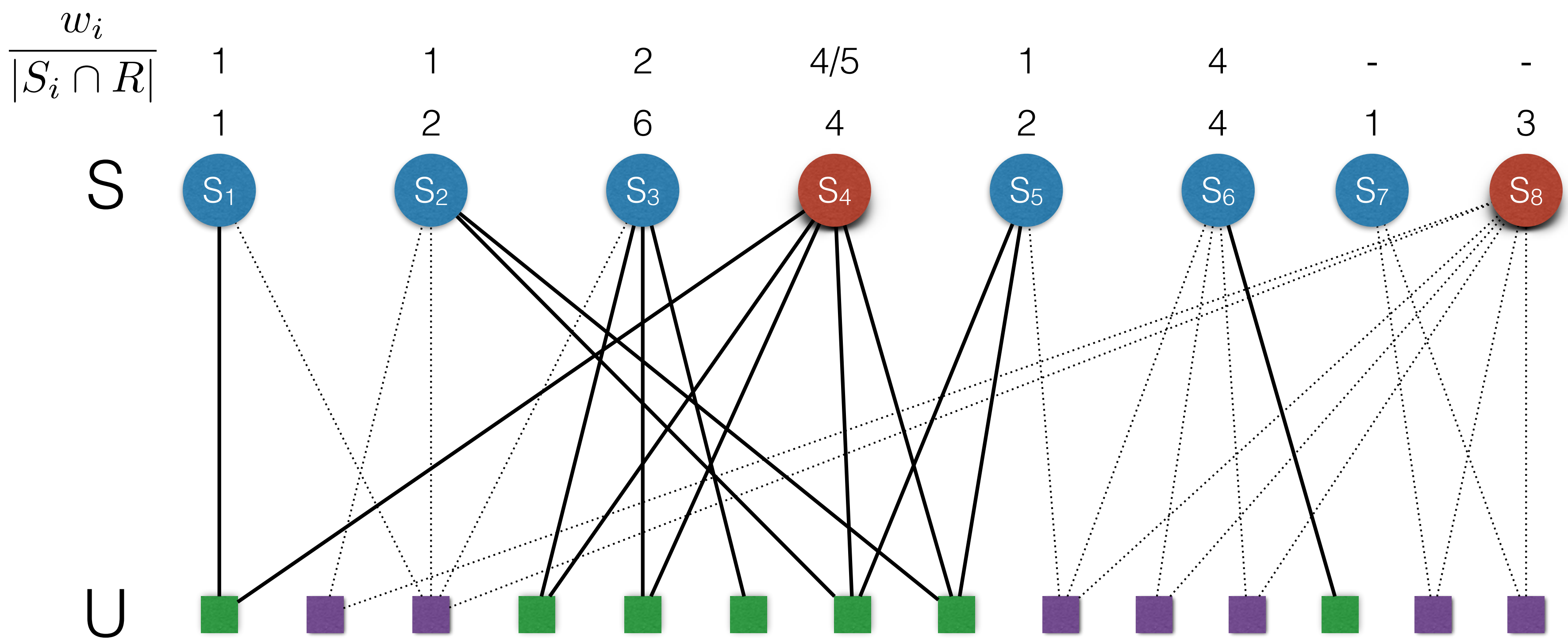

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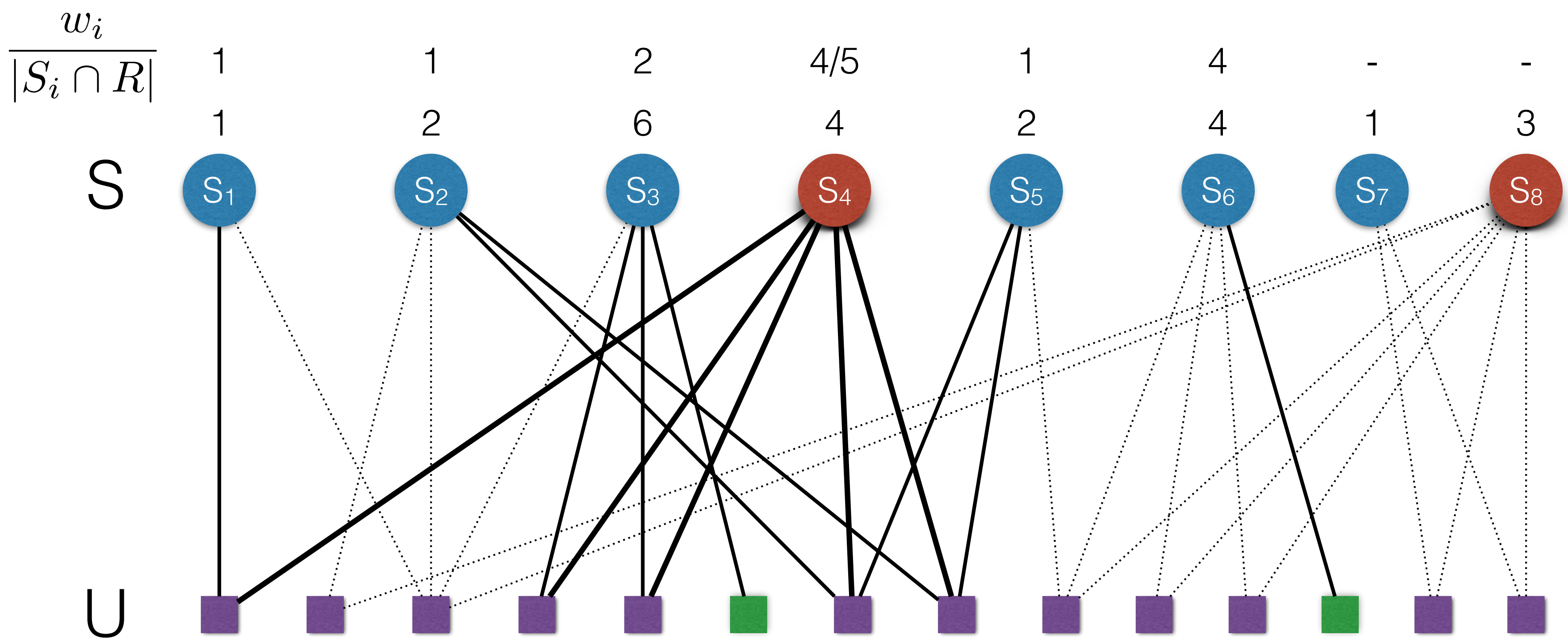

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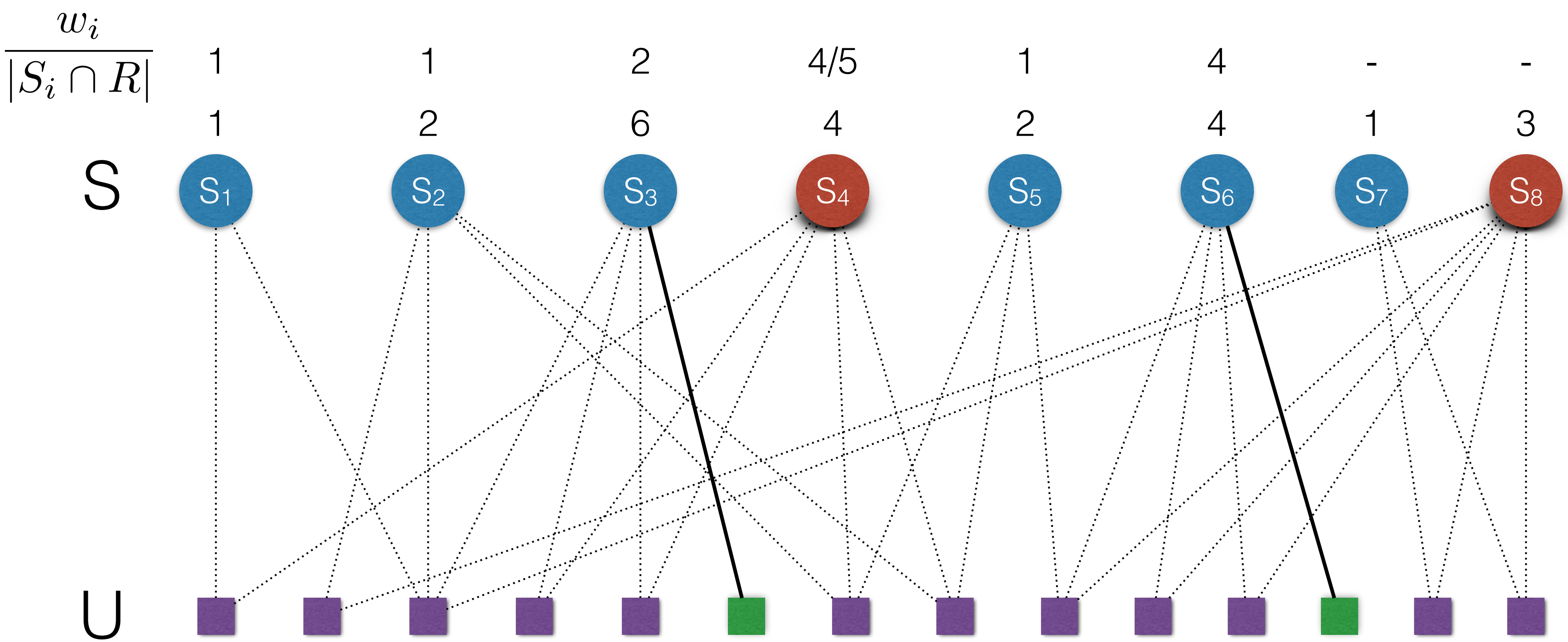

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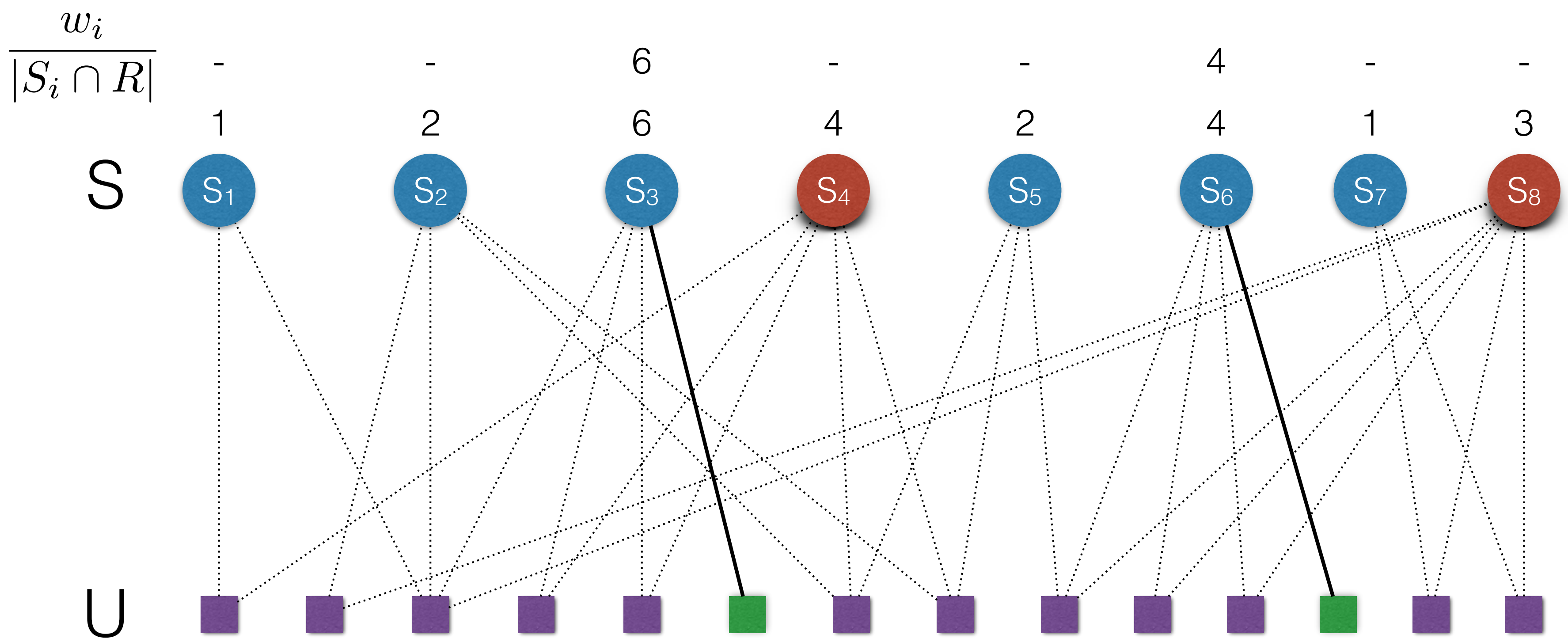

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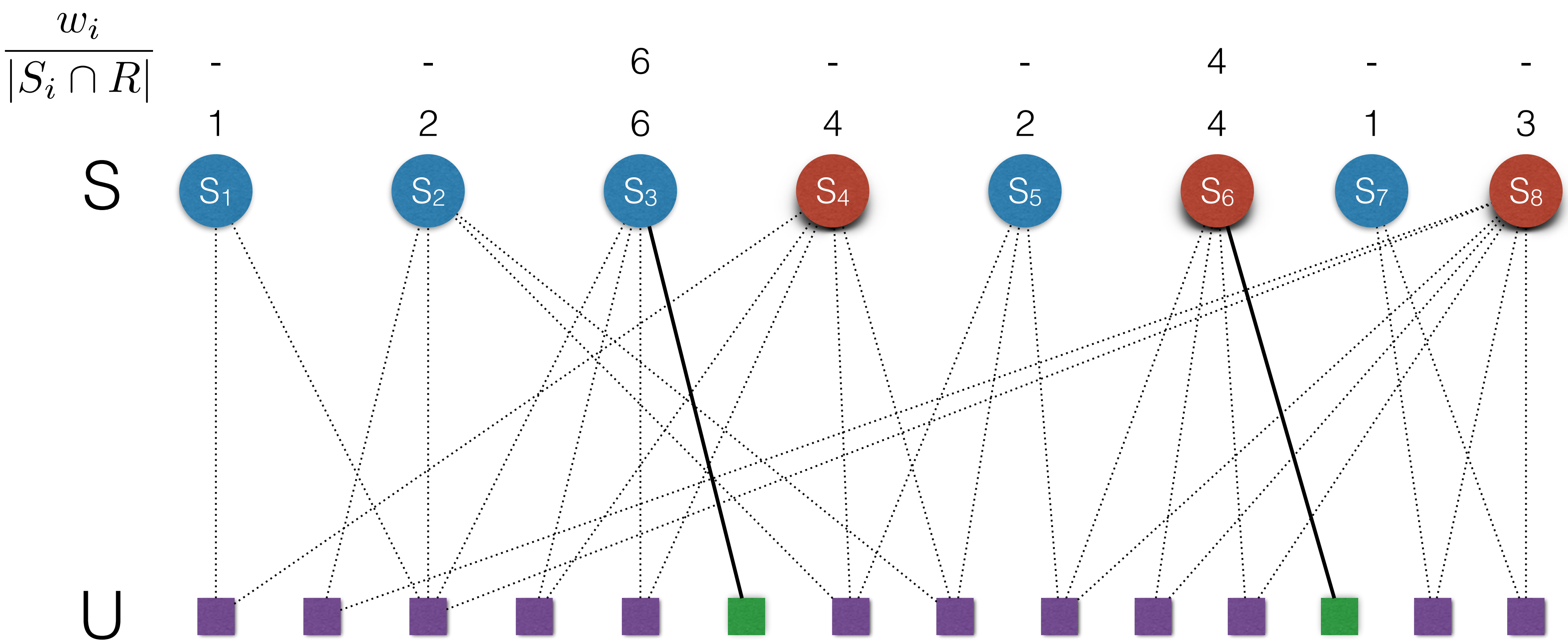

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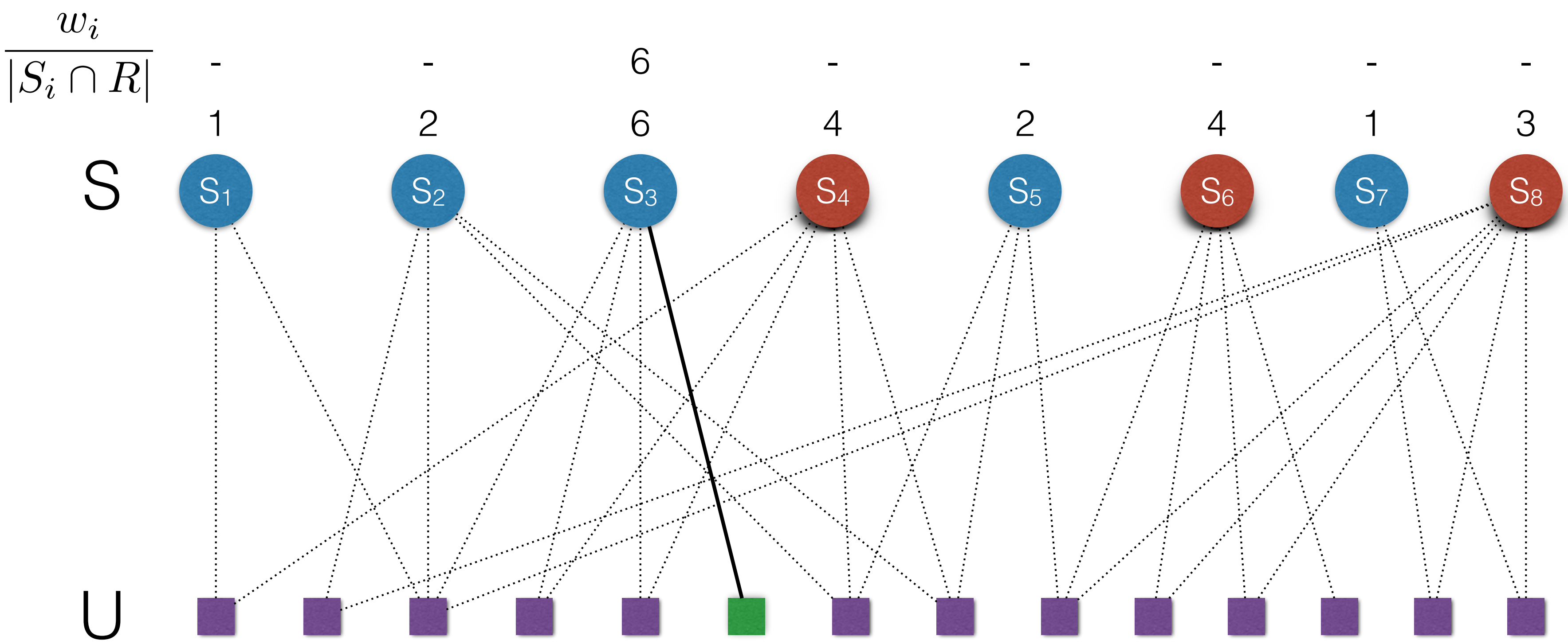

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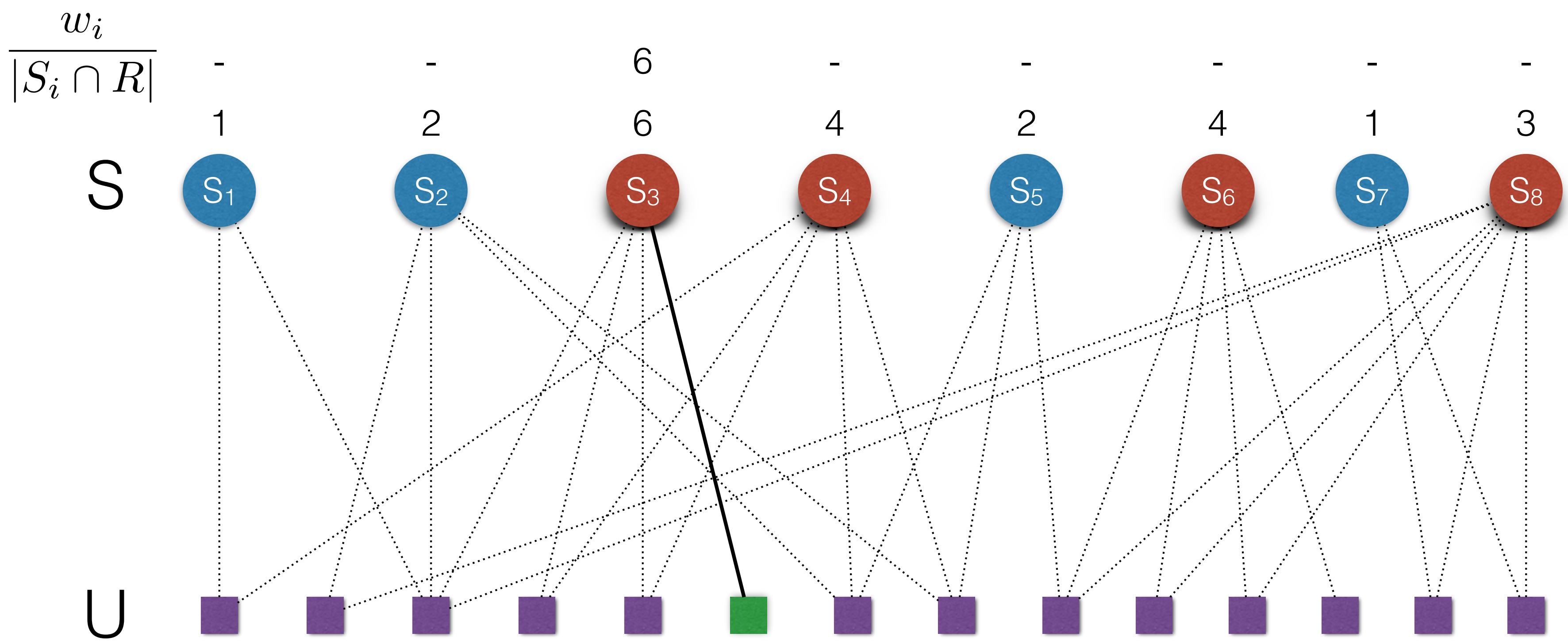

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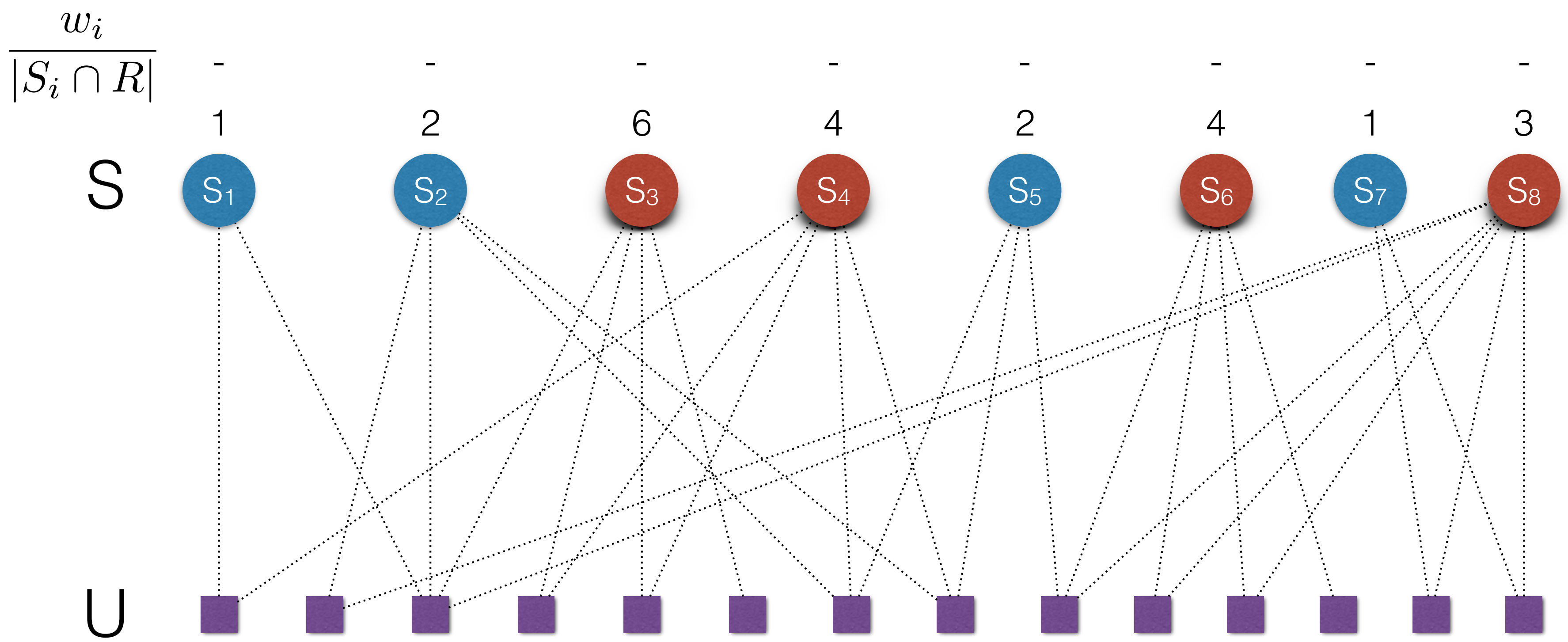


Set cover: greedy algorithm

Greedy-set-cover

Set R := U and C := Ø

while R ≠ Ø

Select the set S_i minimizing $\dfrac{w_i}{|S_i \cap R|}$

Delete the elements from S_i from R.

Add S_i to C.

endwhile

Return C.

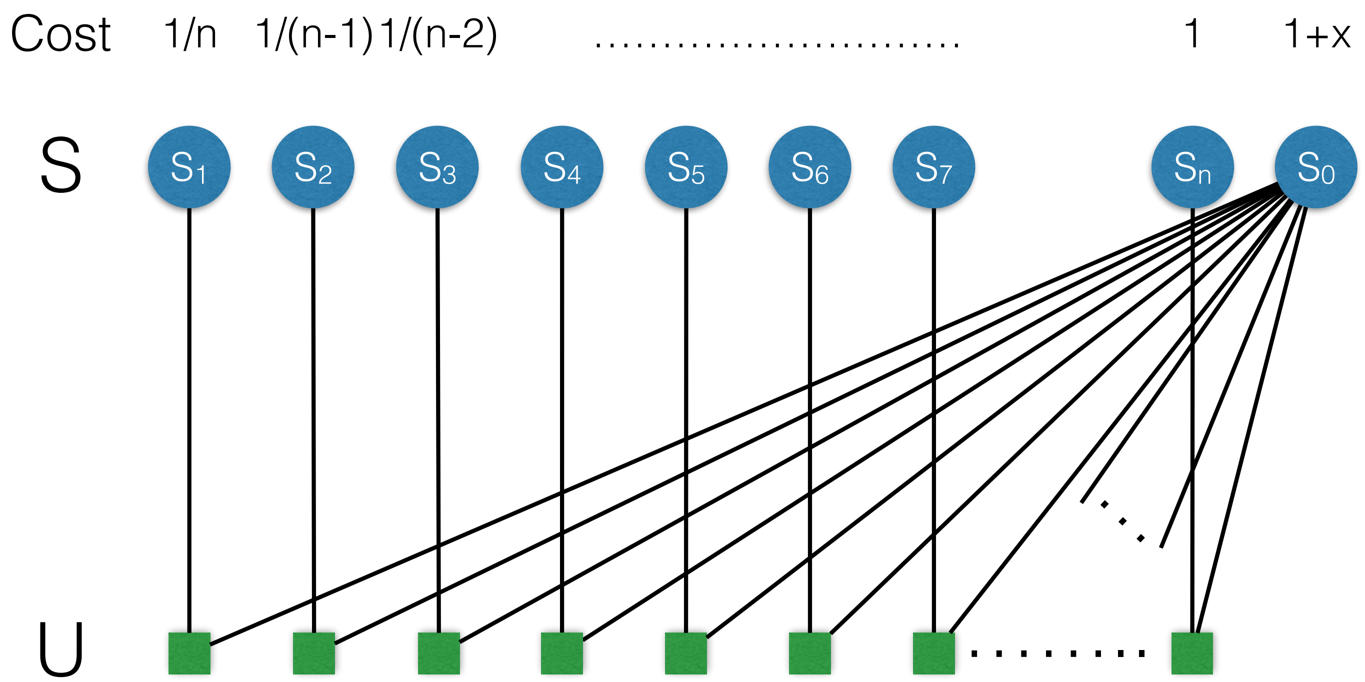
- Greedy-set-cover is an O(log n)-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor O(log n)

Greedy set cover analysis

- $c_e = w_i/(S_i \cap R)$. element e got covered when we picked set i. cost of e is.
- $\text{cost}(C) = \sum_{i \in C} w_i = \sum_{e \in U} c(e)$
- Enumerate elements $e_1, e_2, \ldots, e_n$ in order they were covered.
- Show $c(e_k) \leq \dfrac{OPT}{n-k+1}$
    - e_k covered in round i. Let S_j be set picked in round i.
    - Consider elements R not covered in round i. Optimal alg. can cover R at cost at most OPT, since it can cover all at cost OPT. => in round i OPT can cover R with an average cost per element of OPT/|R|.
    - Must exist a set that has cost efficiency at most average. S_j smallest cost efficiency in round i: $\dfrac{w_j}{|S_j \cap R|} \leq \dfrac{OPT}{|R|}$
    - $|R| \geq n-k+1.$
    - e_k covered by S_j in round i.
        - $c(e_k) = \dfrac{w_j}{|S_j \cap R|} \leq \dfrac{OPT}{|R|} \leq \dfrac{OPT}{n-k+1}$
- Thus
- $\text{cost}(C) = \sum_{e \in U} c(e) \leq \sum_{k=1}^{n} \dfrac{OPT}{n-k+1} = OPT \cdot \sum_{k=1}^{n} \dfrac{1}{n-k+1} = OPT \cdot \left(\dfrac{1}{n} + \dfrac{1}{n-1} + \cdots + 1\right) = OPT \cdot H_n$
- $H_n = O(\log n)$

Greedy set cover analysis:

Set Cover: Greedy algorithm - tight example



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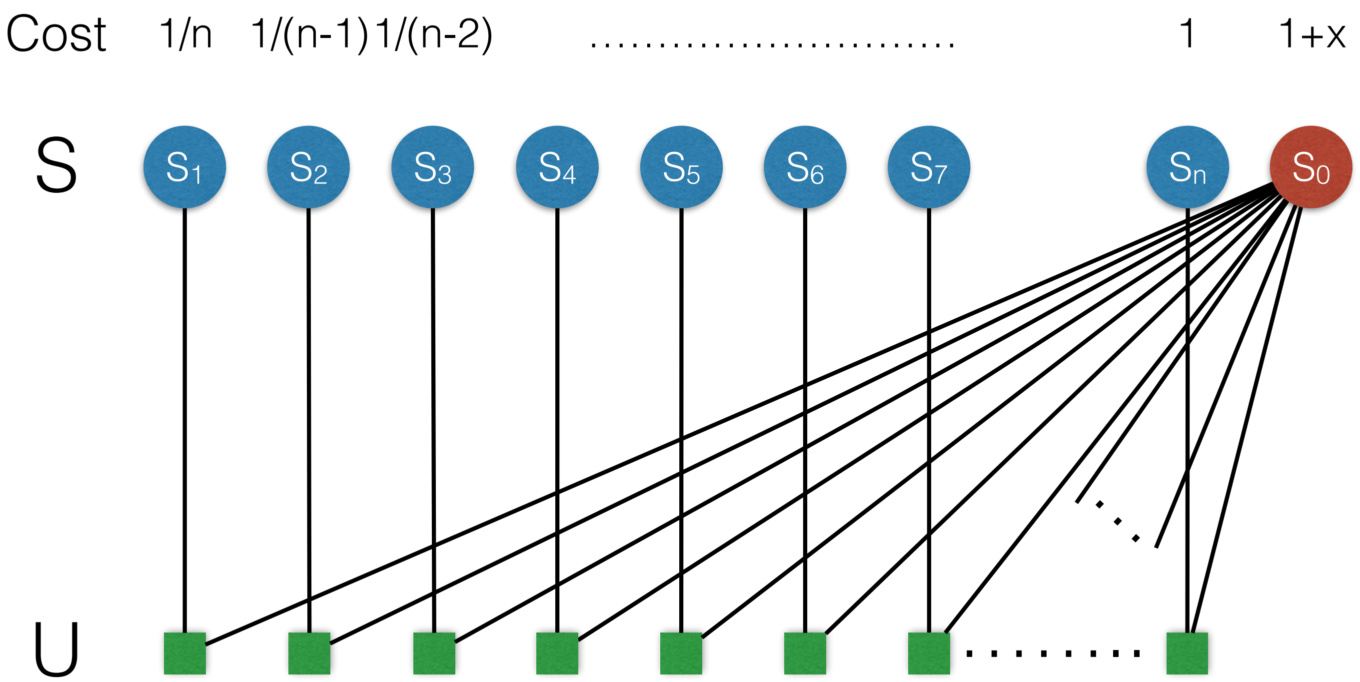
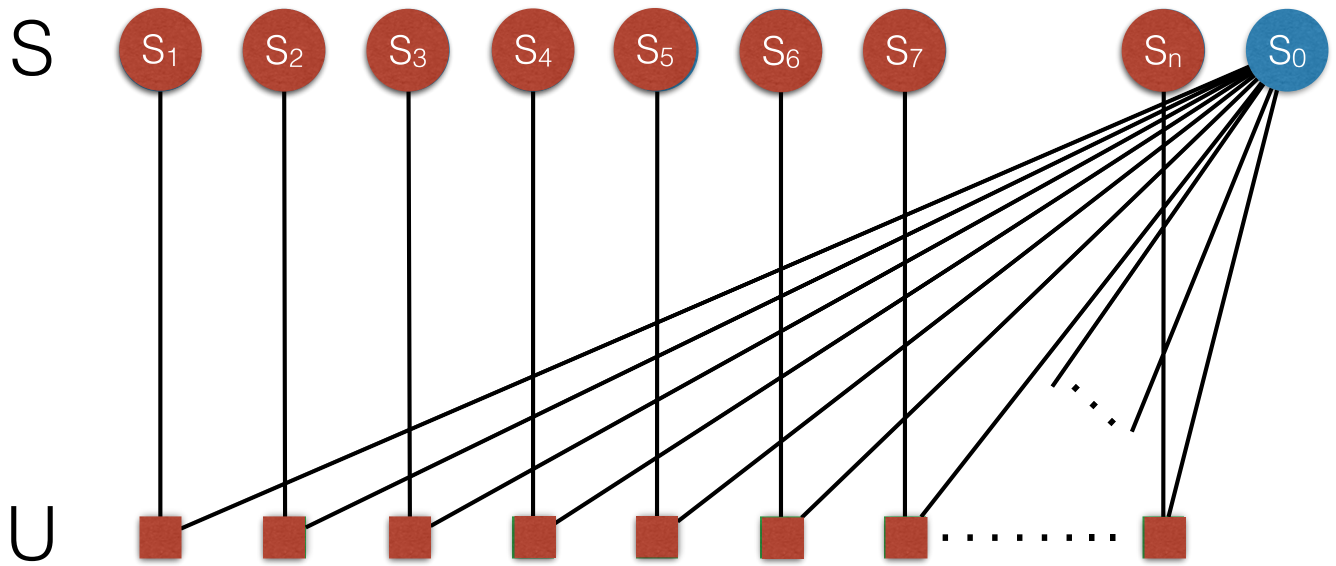


OPT = 1+x

Set Cover: Greedy algorithm - tight example

$\frac{w_i}{|S_i \cap R|}$ 1/n 1/(n-1) 1/(n-2) 1 (1+x)/(n-2)

Cost 1/n 1/(n-1) 1/(n-2) .... 1 1+x



OPT = 1+x

Greedy = 1/n + 1/(n-1) + 1/(n-2) .... = H_n