# Hashing

- Dictionaries
- Chained Hashing
- Universal Hashing
- Static Dictionaries and Perfect Hashing

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## Dictionaries

- Dictionary problem. Maintain a dynamic set of integers $S \subseteq U$ subject to following operations
  - LOOKUP(x): return true if $x \in S$ and false otherwise.
  - INSERT(x): set $S = S \cup \{x\}$
  - DELETE(x): set $S = S \setminus \{x\}$
- Universe size. Typically $|U| = 2^{64}$ or $|U| = 2^{32}$ and $|S| \ll |U|$.
- Satellite information. Information associated with each integer.
- Goal. A compact data structure with fast operations.

## Dictionaries

- Applications.
  - Many!
  - Key component in other data structures and algorithms.

## Dictionaries

- Which solutions do we know?

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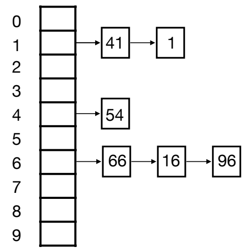
## Chained Hashing

- Chained hashing [Dumey 1956].
  - Hash function. Pick some crazy, chaotic, random function h that maps U to {0, ..., m-1}, where m = Θ(n).
  - Initialize an array A[0, ..., m-1].
  - A[i] stores a linked list containing the keys in S whose hash value is i.

## Chained Hashing

U = {0, ..., 99}
S = {1, 16, 41, 54, 66, 96}
h(x) = x mod 10




- Operations.
  - LOOKUP(x): Compute h(x). Scan through list for h(x). Return true if x is in list and false otherwise.
  - INSERT(x): Compute h(x). Scan through list for h(x). If x is in list do nothing. Otherwise, add x to the front of list.
  - DELETE(x): Compute h(x). Scan through list for h(x). If x is in list remove it. Otherwise, do nothing.
- Time. O(1 + length of linked list for h(x))

## Chained Hashing

- Hash functions.
  - h(x) = x mod 10 is not very crazy, chaotic, or random.
  - For any fixed choice of h, there is a set whose elements all map to the same slot.
  - ⇒ We end up with a single linked list.
  - How can we overcome this?
- Use randomness.
  - Assume the input set is random.
  - Choose the hash function at random.

## Chained Hashing

- Random hash functions. Assume that:
  1. h is chosen uniformly at random among all functions from U to {0,..., m-1}
  2. We can store h in O(n) space.
  3. We can evaluate h in O(1) time
- What is the expected length of the linked lists?

## Chained Hashing

$$
\begin{aligned}
E(\text{length of linked list for } h(x)) &= E\left(|\{y \in S \mid h(y) = h(x)\}|\right) \\
&= E\left(\sum_{y \in S} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} \Pr(h(x) = h(y)) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \Pr(h(x) = h(y))) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} \\
&= 1 + (n-1) \cdot \frac{1}{m} = O(1)
\end{aligned}
$$

$m^2$ choices for pair (h(x), h(y)), m of which cause collision

## Chained Hashing

- Theorem. We can solve the dictionary problem (under assumptions 1+2+3) in
  - O(n) space.
  - O(1) expected time per operation.
- Expectation is over the choice of hash function.
- Independent of the input set.

## Chained Hashing

- Random hash functions assumptions.
  1. h is chosen uniformly at random among all functions from U to {0,…, m-1}
  2. We can store h in O(n) space.
  3. We can evaluate h in O(1) time
- Random hash functions. Can we efficiently compute and store a random function?
  - We need Θ(u log m) bits to store an arbitrary function h: {0,…, u-1} → {0,…, m-1}
  - We need a lot of random bits to generate the function.
  - We need a lot of time to generate the function.
- Do we need a truly random hash function?
- When did we use the fact that h was random in our analysis?

## Chained Hashing

$$
\begin{aligned}
E(\text{length of linked list for } h(x)) &= E\left(|\{y \in S \mid h(y) = h(x)\}|\right) \\
&= E\left(\sum_{y \in S} \begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} E\left(\begin{cases} 1 & \text{if } h(y) = h(x) \\ 0 & \text{if } h(y) \neq h(x) \end{cases}\right) \\
&= \sum_{y \in S} \Pr(h(x) = h(y)) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \Pr(h(x) = h(y))) \\
&= 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} \\
&= 1 + (n-1) \cdot \frac{1}{m} = O(1)
\end{aligned}
$$

For all $x \neq y$, $\Pr(h(x) = h(y)) \leq 1/m$

# Hashing

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- Universal Hashing
- Static Dictionaries and Perfect Hashing

## Universal Hashing

- Universel hashing [Carter and Wegman 1979].
  - Let H be a family of functions mapping U to {0, ..., m-1}.
  - H is universal if for any x≠y in U and h chosen uniformly at random in H,

    Pr(h(x) = h(y)) ≤ 1/m

## Universal Hashing

- Positional number systems. For integers x and m, the base-m representation of x is x written in base m.
- Example.
  - $(10)_{10} = (1010)_2$ $(1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0)$
  - $(107)_{10} = (212)_7$ $(2\cdot 7^2 + 1\cdot 7^1 + 2\cdot 7^0)$

## Universal Hashing

- Hash function. Given a prime m and $a = (a_1a_2\ldots a_r)_m$, define

$$h_a((x_1x_2\ldots x_r)_m) = a_1x_1 + a_2x_2 + \ldots + a_rx_r \bmod m$$

- Example.
  - $m = 7$
  - $a = (107)_{10} = (212)_7$
  - $x = (214)_{10} = (424)_7$
  - $h_a(x) = 2\cdot 4 + 1\cdot 2 + 2\cdot 4 \bmod 7 = 18 \bmod 7 = 4$
- Universal family.
  - $H = \{h_a \mid (a_1a_2\ldots a_r)_m \in \{0, \ldots, m-1\}^r\}$
  - Choose random hash function from H ~ choose random a.
  - H is universal (analysis next).
  - O(1) time evaluation.
  - O(1) space.
  - Fast construction.

## Universal Hashing

- Lemma. Let m be a prime. For any $a \in \{1, \ldots, m-1\}$ there exists a unique inverse $a^{-1}$ such that $a^{-1} \cdot a \equiv 1 \mod m$. ($Z_m$ is a field)
- Example. m = 7

| a | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a^{-1}$ | | | | | | |

| a | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $a^{-1}$ | 1 | 4 | 5 | 2 | 3 | 6 |

## Universal Hashing

- Goal. For random $a = (a_1a_2\ldots a_r)_m$, show that if $x = (x_1x_2\ldots x_r)_m \neq y = (y_1y_2\ldots y_r)_m$ then $\Pr[h_a(x) = h_a(y)] \leq 1/m$
- $(x_1x_2\ldots x_r)_m \neq (y_1y_2\ldots y_r)_m \Longrightarrow x_i \neq y_i$ for some i. Assume wlog. that $x_r \neq y_r$.

$$\begin{aligned}
&\Pr(h_a((x_1\ldots x_r)_m) = h_a((y_1\ldots,y_r)_m)) \\
&= \Pr\left(a_1x_1 + \cdots + a_rx_r \equiv a_1y_1 + \cdots + a_ry_r \bmod m\right) \\
&= \Pr\left(a_rx_r - a_ry_r \equiv a_1y_1 - a_1x_1 + \cdots + a_{r-1}y_{r-1} - a_{r-1}x_{r-1} \bmod m\right) \\
&= \Pr\left(a_r(x_r - y_r) \equiv a_1(y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}) \bmod m\right) \\
&= \Pr\left(a_r(x_r - y_r)(x_r - y_r)^{-1} \equiv (a_1(y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}))(x_r - y_r)^{-1} \bmod m\right) \\
&= \Pr\left(a_r \equiv (a_1(y_1 - x_1) + \cdots + a_{r-1}(y_{r-1} - x_{r-1}))(x_r - y_r)^{-1} \bmod m\right) = \frac{1}{m}
\end{aligned}$$

existence of inverses

for any choice of $a_1, a_2, \ldots, a_{r-1}$, the RH defines a unique $a_r$ that matches (uniqueness of inverses).
Of the $m^r$ choices for $a_1, a_2, \ldots, a_r$ exactly $m^{r-1}$ cause a collision ⇒ probability is $m^{r-1}/m^r = 1/m$

## Universal Hashing

- Lemma. H is universal with O(1) time evaluation and O(1) space.

- Theorem. We can solve the dictionary problem (without special assumptions) in:
  - O(n) space.
  - O(1) expected time per operation (lookup, insert, delete).

## Universal Hashing

- Other universal families.
  - For prime p > 0, a ∈ {1, .., p-1}, b ∈ {0, ..., p-1}

$$h_{a,b}(x) = ax + b \bmod m$$
$$H = \{h_{a,b} \mid a \in \{1, \ldots, m-1\}, b \in \{0, \ldots, m-1\}\}$$

  - Hash function from k-bit numbers to l-bit numbers. a is an odd k-bit integer.

l most significant bits of the k least significant bits of ax

$$h_a(x) = (ax \bmod 2^k) \gg (k - l)$$
$$H = \{h_a \mid a \text{ is an odd integer in } \{1, \ldots, 2^k - 1\}\}$$

# Hashing

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## Static Dictionaries and Perfect Hashing

- Static dictionary problem. Given a set S ⊆ U = {0,..,u-1} of size n for preprocessing support the following operation
  - lookup(x): return true if x ∈ S and false otherwise.

- As the dictionary problem with no updates (insert and deletes).
- Set given in advance.

## Static Dictionaries and Perfect Hashing

- Dynamic solution. Use chained hashing with a universal hash function as before $\Longrightarrow$ solution with O(n) space and O(1) expected time per lookup.
- Can we do better?
- Perfect Hashing. A perfect hash function for S is a collision-free hash function on S.
  - Perfect hash function in O(n) space and O(1) evaluation time $\Longrightarrow$ solution with O(n) space and O(1) worst-case lookup time.
  - Do perfect hash functions with O(n) space and O(1) evaluation time exist for any set S?

## Static Dictionaries and Perfect Hashing

- Goal. Perfect hashing in linear space and constant worst-case time.
- Solution in 3 steps.
  - Solution 1. Collision-free but with too much space.
  - Solution 2. Many collisions but linear space.
  - Solution 3: FKS scheme [Fredman, Komlós, Szemerédi 1984]. Two-level solution. Combines solution 1 and 2.
    - At level 1 use solution with lots of collisions and linear space.
    - Resolve collisions at level 1 with collision-free solution at level 2.
    - lookup(x): look-up in level 1 to find the correct level 2 dictionary. Lookup in level 2 dictionary.

## Static Dictionaries and Perfect Hashing

- Solution 1. Collision-free but with too much space.
- Use a universal hash function to map into an array of size $n^2$. What is the expected total number of collisions in the array?

$$
\begin{aligned}
E(\#\text{collisions}) &= E\left(\sum_{x,y\in S, x\neq y} \begin{cases} 1 & \text{if } h(y)=h(x) \\ 0 & \text{if } h(y)\neq h(x) \end{cases}\right) \\
&= \sum_{x,y\in S, x\neq y} E\left(\begin{cases} 1 & \text{if } h(y)=h(x) \\ 0 & \text{if } h(y)\neq h(x) \end{cases}\right) \\
&= \sum_{x,y\in S, x\neq y} \Pr(h(x)=h(y)) = \binom{n}{2}\frac{1}{n^2} \leq \frac{n^2}{2}\cdot\frac{1}{n^2} = 1/2
\end{aligned}
$$

#distinct pairs — universal hashing into $n^2$ range

- With probability 1/2 we get perfect hashing function. If not perfect try again.
- $\Longrightarrow$ Expected number of trials before we get a perfect hash function is O(1).
- $\Longrightarrow$ For a static set S we can support lookups in O(1) worst-case time using O($n^2$) space.

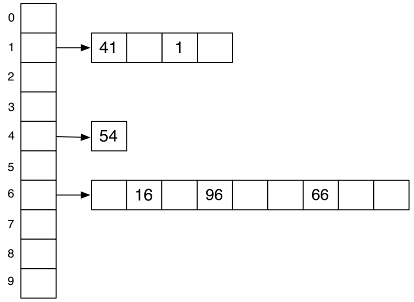
## Static Dictionaries and Perfect Hashing

- Solution 2. Many collisions but linear space.
- As solution 1 but with array of size n. What is the expected total number of collisions in the array?

$$
\begin{aligned}
E(\#\text{collisions}) &= E\left(\sum_{x,y\in S, x\neq y} \begin{cases} 1 & \text{if } h(y)=h(x) \\ 0 & \text{if } h(y)\neq h(x) \end{cases}\right) \\
&= \sum_{x,y\in S, x\neq y} E\left(\begin{cases} 1 & \text{if } h(y)=h(x) \\ 0 & \text{if } h(y)\neq h(x) \end{cases}\right) \\
&= \sum_{x,y\in S, x\neq y} \Pr(h(x)=h(y)) = \binom{n}{2}\frac{1}{n} \leq \frac{n^2}{2}\cdot\frac{1}{n} = \frac{1}{2}n
\end{aligned}
$$

## Static Dictionaries and Perfect Hashing

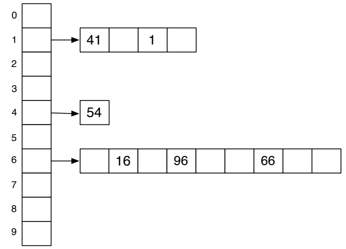
- Solution 3. Two-level solution.
  - At level 1 use solution with lots of collisions and linear space.
  - Resolve each collisions at level 1 with collision-free solution at level 2.
  - lookup(x): look-up in level 1 to find the correct level 2 dictionary. Lookup in level 2 dictionary.
- Example.
  - S = {1, 16, 41, 54, 66, 96}
  - Level 1 collision sets:
    - $S_1$ = {1, 41},
    - $S_4$ = {54},
    - $S_6$ = {16, 66, 96}
  - Level 2 hash info stored with subtable.
    - (size of table, multiplier a, prime p)
- Time. O(1)
- Space?




## Static Dictionaries and Perfect Hashing

- Space. What is the the total size of level 1 and level 2 hash tables?

$$\#\text{collisions} = \sum \binom{|S_i|}{2} = O(n)$$




$a^2 = a + 2\binom{a}{2}$, for any integer $a$

$$\text{space} = O\left(n + \sum_i |S_i|^2\right) = O\left(n + \sum_i \left(|S_i| + 2\binom{|S_i|}{2}\right)\right)$$

$$= O\left(n + \sum_i |S_i| + 2\sum_i \binom{|S_i|}{2}\right) = O(n + n + 2n) = O(n)$$

## Static Dictionaries and Perfect Hashing

- FKS scheme.
  - O(n) space and O(n) expected preprocessing time.
  - Lookups with two evaluations of a universal hash function.
- Theorem. We can solve the static dictionary problem for a set S of size n in:
  - O(n) space and O(n) expected preprocessing time.
  - O(1) worst-case time per lookup.
- Multilevel data structures.
  - FKS is example of multilevel data structure technique. Combine different solutions for same problem to get an improved solution.

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