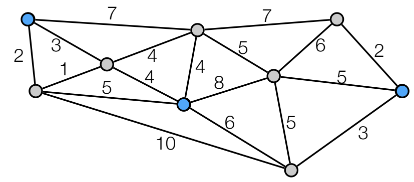
k-center

The k-center problem

- **Input**. An integer k and a complete, undirected graph G=(V,E), with distance d(i,j) between each pair of vertices i,j ∈ V.
- d is a metric:
  - dist(i,i) = 0
  - dist(i,j) = dist(j,i)
  - dist(i,l) ≤ dist(i,j) + dist(j,l)
- **Goal.** Choose a set S ⊆ V , |S| = k, of k centers so as to minimize the maximum distance of a vertex to its closest center.

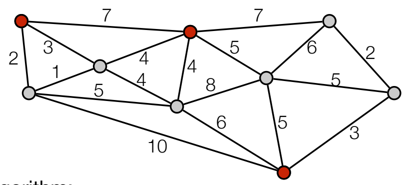
$$S = \text{argmin}_{S \subseteq V, |S|=k} \max_{i \in V} \text{dist}(i,S)$$

- **Covering radius.** Maximum distance of a vertex to its closest center.



k-center: Greedy algorithm

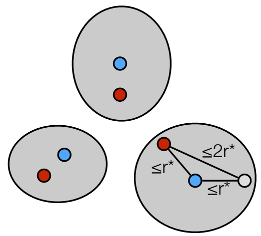
- Greedy algorithm.
  - Pick arbitrary i in V.
  - Set S = {i}
  - while |S| < k do
    - Find vertex j farthest away from any cluster center in S
    - Add j to S



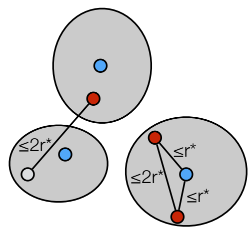
- Greedy is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

k-center: analysis greedy algorithm

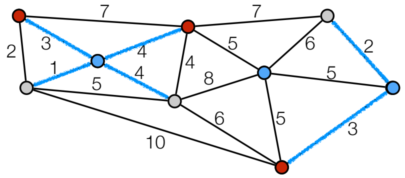
- r* optimal radius.
- Show all vertices within distance 2r* from a center.
- Consider optimal clusters. 2 cases.
  - Algorithm picked one center in each optimal cluster
    - distance from any vertex to its closest center ≤ 2r* (triangle inequality)



  - Some optimal cluster does not have a center.
    - Some cluster have more than one center.
    - distance between these two centers ≤ 2r*.
    - when second center in same cluster picked it was the vertex farthest away from any center.
    - distance from any vertex to its closest center at most 2r*.

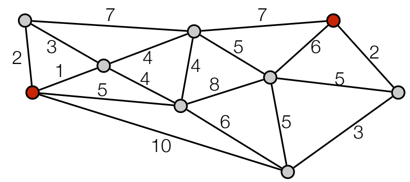


k-center



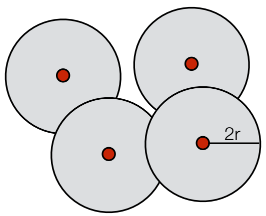
Bottleneck algorithm

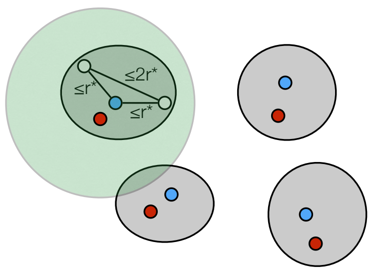
- Assume we know the optimum covering radius r.
- Bottleneck algorithm.
  - Set R := V and S := Ø.
  - while R ≠ Ø do
    - Pick arbitrary i in R.
    - Add j to S
    - Remove all vertices with d(j,v) ≤ 2r from R.
- Example: k= 3. r = 4.



Analysis bottleneck algorithm

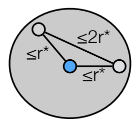
- r* optimal radius.
- Covering radius is at most 2r*.
- Show that: We cannot pick more than k centers:
  - We can pick at most one in each optimal cluster:
    - Distance between two nodes in same optimal cluster ≤ 2r.*
    - When we pick a center in a optimal cluster all nodes in same optimal cluster is removed.

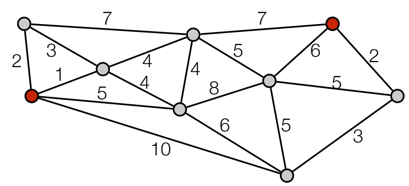




Bottleneck algorithm

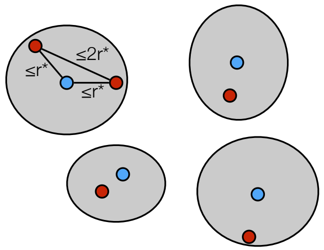
- Assume we know the optimum covering radius r.
- Example: k= 3. r = 4.
- Analysis.
  - Covering radius is at most 2.
  - Algorithm picks more than k centers => the optimum covering radius is > r.
    - If algorithm pick more than k centers then it picked more than one in some OPT cluster.
    - If r* ≤ r we can pick at most one in each optimum cluster.
- Can “guess” optimal covering radius (only a polynomial number of possible values).





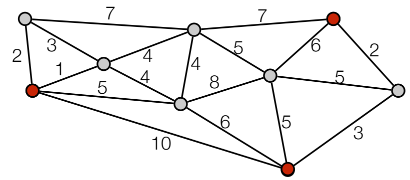
Analysis bottleneck algorithm

- r^* optimal radius.
- Can use algorithm to “guess” r^* (at most n^2 values).
- If algorithm picked more than k centers then $r^* > r$.
  - If algorithm picked more than k centers then it picked more than one in some optimal cluster.
  - Distance between two nodes in same optimal cluster $\leq 2r^*$.
  - If more than one in some optimal cluster then $2r < 2r^*$.



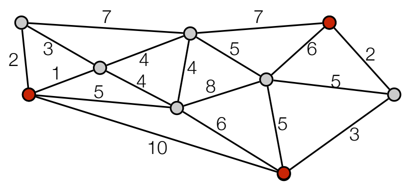
Bottleneck algorithm

- Assume we don’t know the optimum covering radius r.
- Example: k= 3.
- Try with r=2:
  - Still vertices left after picking 3 centers => $r^* > 2$.



Bottleneck algorithm

- Assume we don’t know the optimum covering radius r.
- Example: k= 3.
- Try with r=3:



- All vertices deleted after picking 3 centers
- Know $r^* \geq 3$ (from last round).
- Max distance from a vertex to a center is $2r = 6 \leq 2r^*$.

k-center: Inapproximability

- There is no α-approximation algorithm for the k-center problem for $\alpha < 2$ unless P=NP.
- **Proof.** Reduction from dominating set.
- *Dominating set.* Given G=(V,E) and k. Is there a (dominating) set $S \subseteq V$ of size k, such that each vertex is either in S or adjacent to a vertex in S?
- Given instance of the dominating set problem construct instance of k-center problem:
  - Complete graph G’ on V.
  - All edges from E has weight 1, all new edges have weight 2.
  - Radius in k-center instance 1 or 2.
  - G has an dominating set of size k <=> opt solution to the k-center problem has radius 1.
  - Use α-approximation algorithm A:
    - opt = 1 => A returns solution with radius at most $\alpha < 2$.
    - opt = 2 => A returns solution with radius at least 2.
    - Can use A to distinguish between the 2 cases.