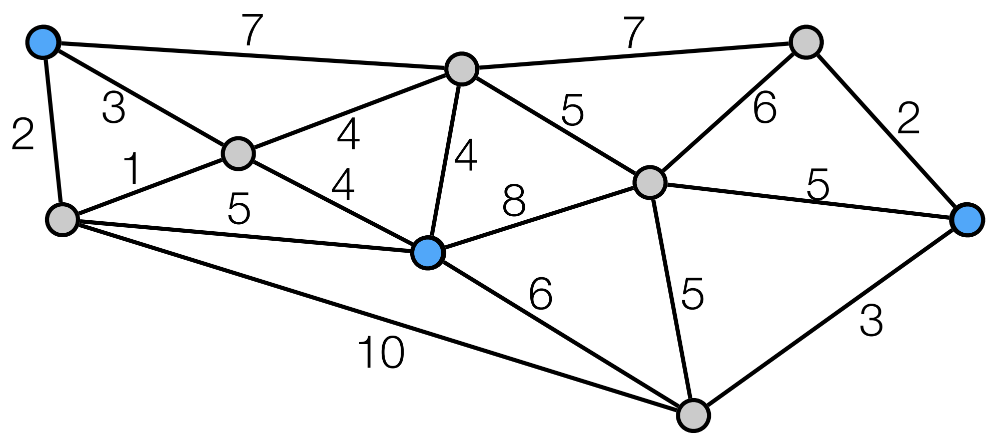
# k-center

# The k-center problem

- **Input**. An integer k and a complete, undirected graph $G=(V,E)$, with distance $d(i,j)$ between each pair of vertices $i,j \in V$.
- d is a metric:
  - $dist(i,i) = 0$
  - $dist(i,j) = dist(j,i)$
  - $dist(i,l) \leq dist(i,j) + dist(j,l)$
- **Goal.** Choose a set $S \subseteq V$ , $|S| = k$, of k centers so as to minimize the maximum distance of a vertex to its closest center.

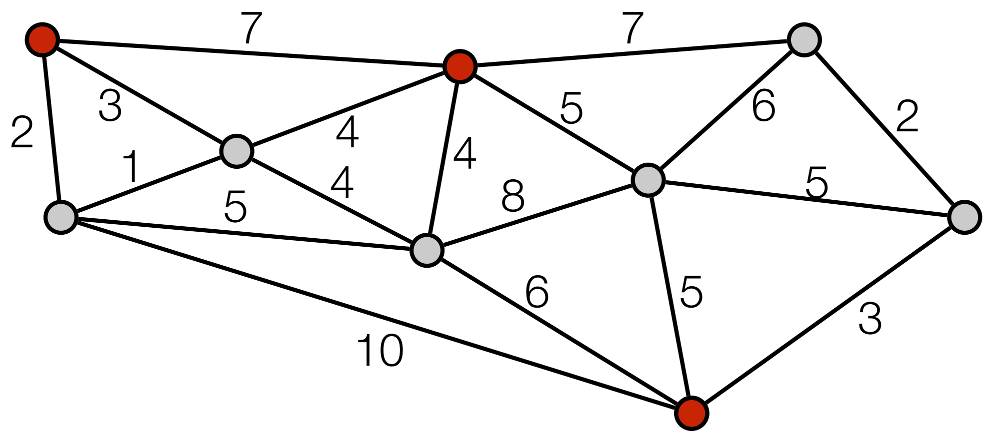
$$S = \text{argmin}_{S \subseteq V, |S|=k} \max_{i \in V} dist(i,S)$$

- **Covering radius.** Maximum distance of a vertex to its closest center.



# k-center: Greedy algorithm

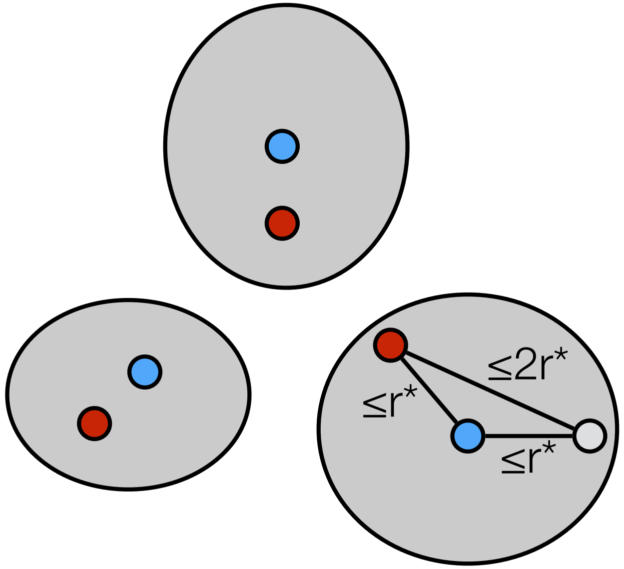
- Greedy algorithm.
  - Pick arbitrary i in V.
  - Set S = {i}
  - while |S| < k do
    - Find vertex j farthest away from any cluster center in S
    - Add j to S



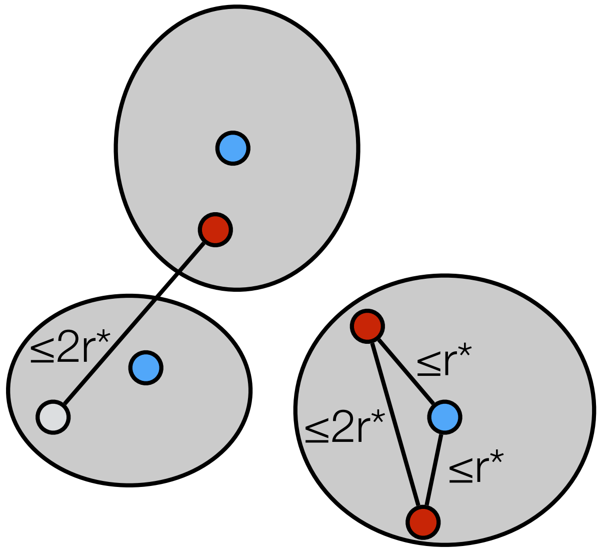

- Greedy is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

# k-center: analysis greedy algorithm

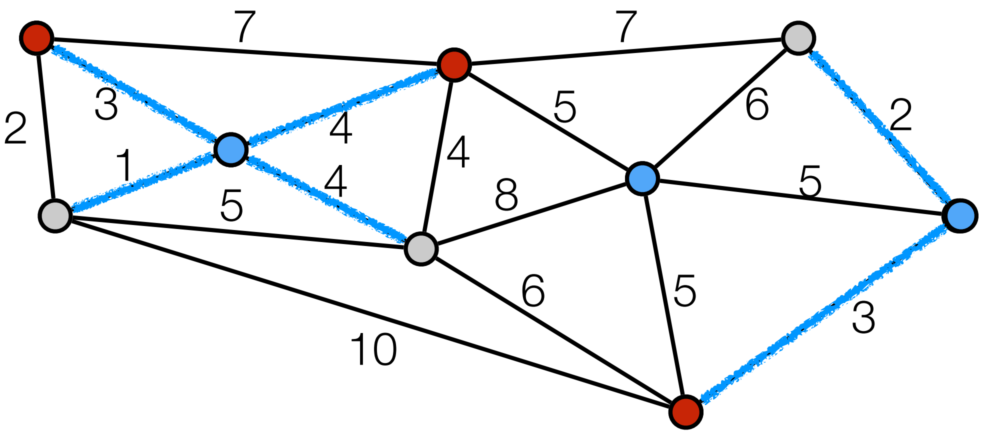
- $r^*$ optimal radius.
- Show all vertices within distance $2r^*$ from a center.
- Consider optimal clusters. 2 cases.
  - Algorithm picked one center in each optimal cluster
    - distance from any vertex to its closest center $\leq 2r^*$ (triangle inequality)




  - Some optimal cluster does not have a center.
    - Some cluster have more than one center.
    - distance between these two centers $\leq 2r^*$.
    - when second center in same cluster picked it was the vertex farthest away from any center.
    - distance from any vertex to its closest center at most $2r^*$.



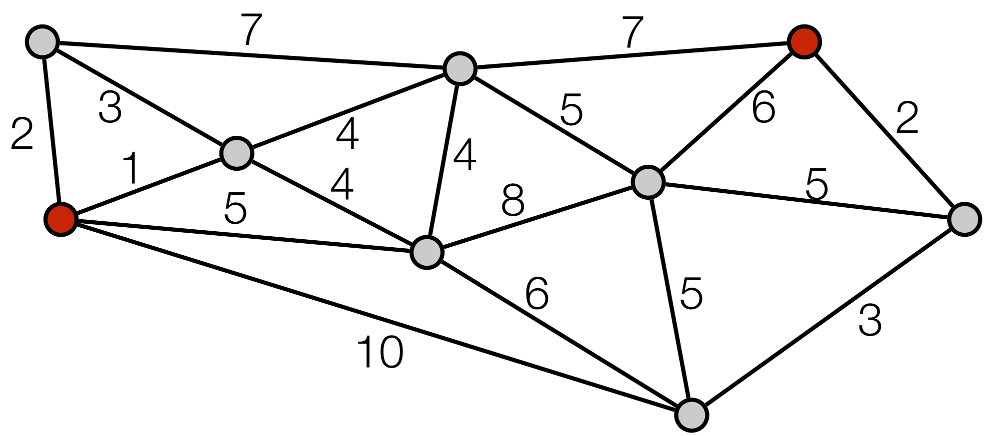
# k-center



# Bottleneck algorithm

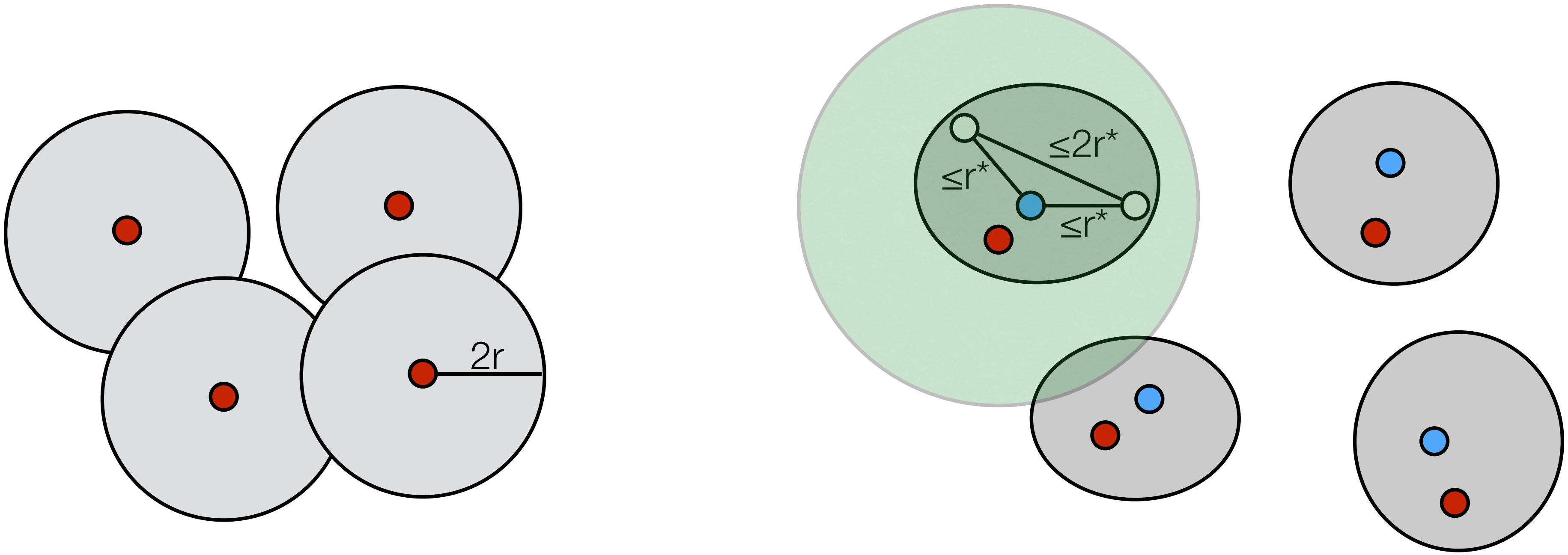
- Assume we know the optimum covering radius r.
- Bottleneck algorithm.
  - Set R := V and S := Ø.
  - while R ≠ Ø do
    - Pick arbitrary i in R.
    - Add j to S
    - Remove all vertices with d(j,v) ≤ 2r from R.

- Example: k= 3. r = 4.



# Analysis bottleneck algorithm

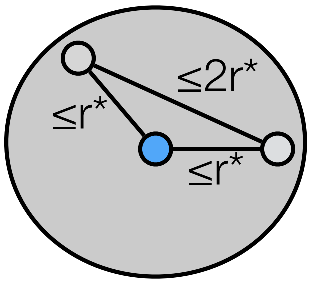
- $r^*$ optimal radius.
- Covering radius is at most $2r^*$.
- Show that: We cannot pick more than k centers:
    - We can pick at most one in each optimal cluster:
        - Distance between two nodes in same optimal cluster $\leq 2r^*$.
        - When we pick a center in a optimal cluster all nodes in same optimal cluster is removed.

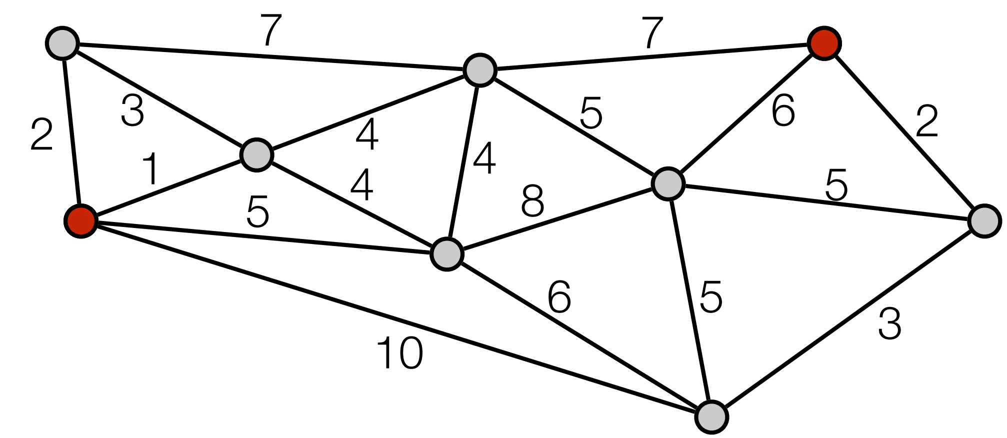


# Bottleneck algorithm

- Assume we know the optimum covering radius r.
- Example: k= 3. r = 4.
- Analysis.
  - Covering radius is at most 2.
  - Algorithm picks more than k centers => the optimum covering radius is > r.
    - If algorithm pick more than k centers then it picked more than one in some OPT cluster.
    - If r* ≤ r we can pick at most one in each optimum cluster.

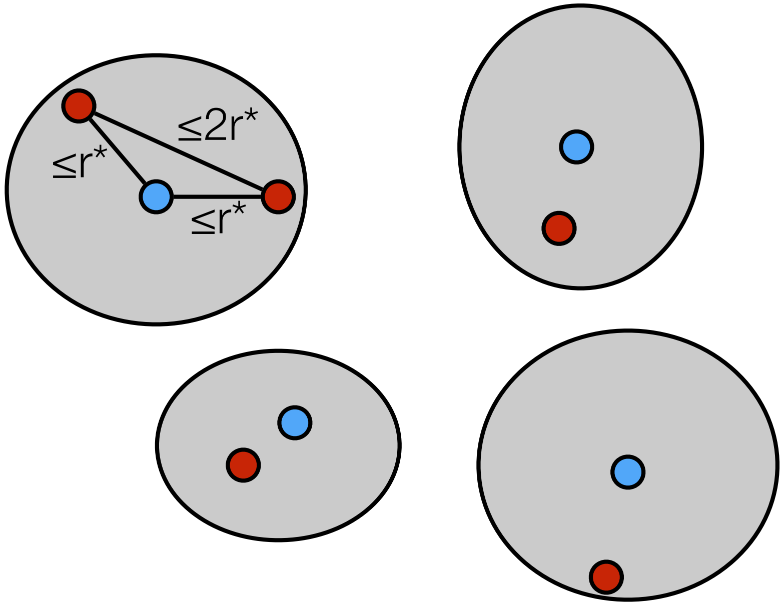
- Can "guess" optimal covering radius (only a polynomial number of possible values).



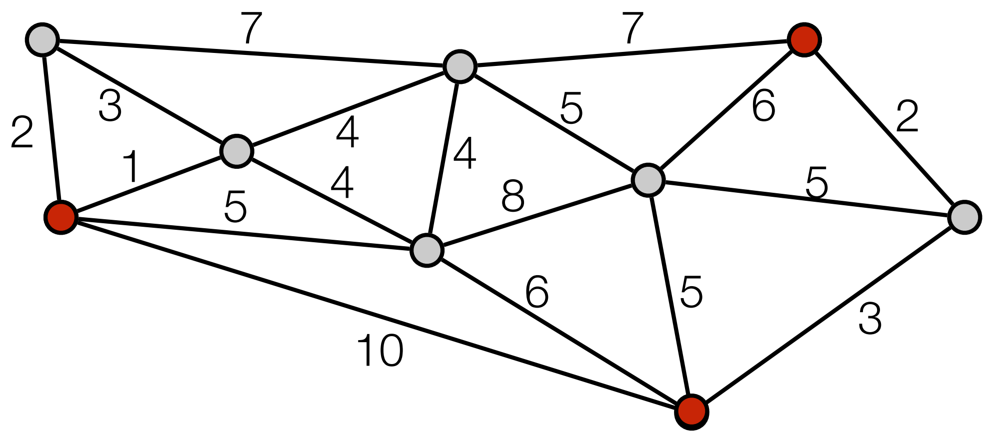
# Analysis bottleneck algorithm

- $r^*$ optimal radius.
- Can use algorithm to "guess" $r^*$ (at most $n^2$ values).
- If algorithm picked more than k centers then $r^* > r$.
    - If algorithm picked more than k centers then it picked more than one in some optimal cluster.
    - Distance between two nodes in same optimal cluster $\leq 2r^*$.
    - If more than one in some optimal cluster then $2r < 2r^*$.



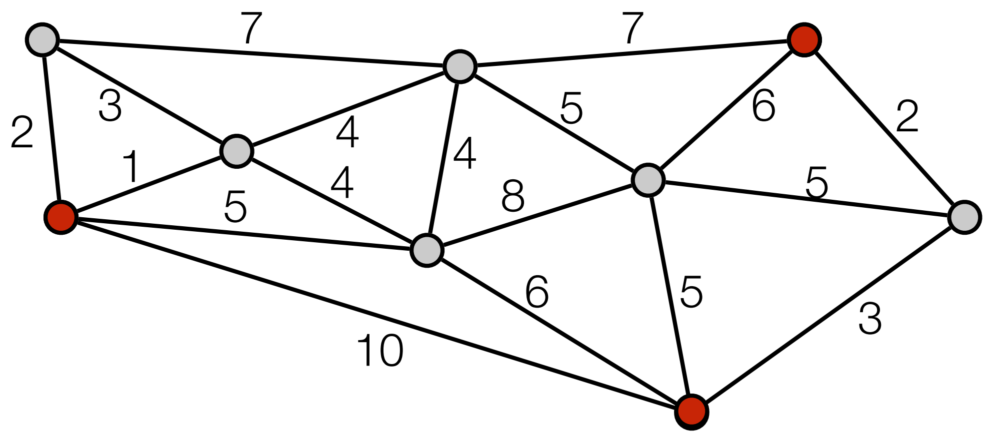
# Bottleneck algorithm

- Assume we don't know the optimum covering radius r.
- Example: k= 3.
- Try with r=2:
    - Still vertices left after picking 3 centers => r* > 2.



# Bottleneck algorithm

- Assume we don't know the optimum covering radius r.
- Example: k= 3.
- Try with r=3:




- All vertices deleted after picking 3 centers
- Know r* ≥ 3 (from last round).
- Max distance from a vertex to a center is 2r = 6 ≤ 2r*.

# k-center: Inapproximability

- There is no α-approximation algorithm for the k-center problem for $\alpha < 2$ unless P=NP.
- **Proof.** Reduction from dominating set.
- *Dominating set.* Given G=(V,E) and k. Is there a (dominating) set $S \subseteq V$ of size k, such that each vertex is either in S or adjacent to a vertex in S?
- Given instance of the dominating set problem construct instance of k-center problem:
    - Complete graph G' on V.
    - All edges from E has weight 1, all new edges have weight 2.
    - Radius in k-center instance 1 or 2.
    - G has an dominating set of size k <=> opt solution to the k-center problem has radius 1.
    - Use α-approximation algorithm A:
        - opt = 1 => A returns solution with radius at most $\alpha < 2$.
        - opt = 2 => A returns solution with radius at least 2.
        - Can use A to distinguish between the 2 cases.