Predecessor

- Predecessor Problem
- van Emde Boas
- Tries

Philip Bille

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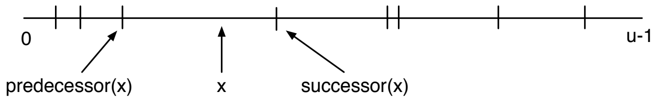
Predecessors

- Predecessor problem. Maintain a set $S \subseteq U = \{0, ..., u-1\}$ supporting
  - predecessor(x): return the largest element in S that is $\leq$ x.
  - sucessor(x): return the smallest element in S that is $\geq$ x.
  - insert(x): set $S = S \cup \{x\}$
  - delete(x): set $S = S - \{x\}$



Predecessors

- Applications.
  - Simplest version of nearest neighbor problem.
  - Several applications in other algorithms and data structures.
  - Probably most practically solved problem in the world: Out all computational resources globally a huge fraction is used to solve the predecessor problem!

Predecessors

- Routing IP-Packets
  - Where should we forward the packet to?
  - To address matching the longest prefix of 192.110.144.123.
  - Equivalent to predecessor problem.
  - Best practical solutions based on advanced predecessor data structures [Degermark, Brodnik, Carlsson, Pink 1997]

Predecessors

- Which solutions do we know?
  - Linked list
  - Balanced binary search trees.
  - Bitvectors

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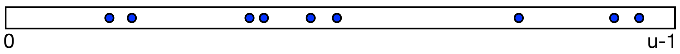
van Emde Boas

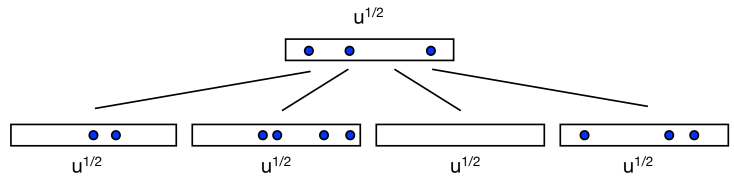
- Goal. Static predecessor with O(log log u) query time.
- Solution in 5 steps.
  - Bitvector. Very slow
  - Two-level bitvector. Slow.
  - ....
  - van Emde Boas [Boas 1975]. Fast.

Solution 1: Bitvector



- Data structure. Bitvector.
- Predecessor(x): Walk left.
- Time. O(u)

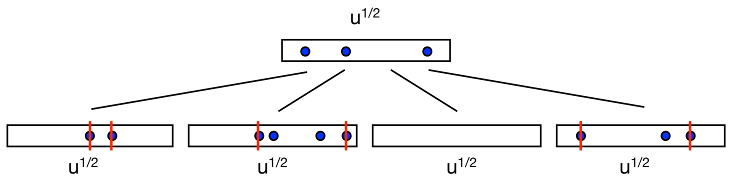
Solution 2: Two-Level Bitvector



- Data structure. Top bitvector + $u^{1/2}$ bottom bitvectors.
- Predecessor(x): Walk left in bottom + walk left in top + walk left bottom.
- Time. $O(u^{1/2} + u^{1/2} + u^{1/2}) = O(u^{1/2})$

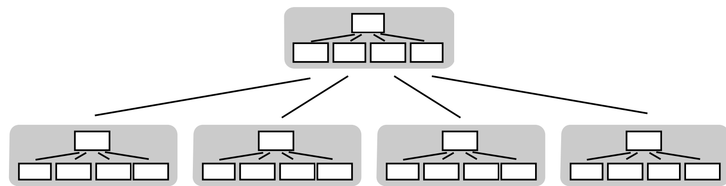
- To find indices in top and bottom write $x = hi(x) \cdot u^{1/2} + lo(w)$
- Index in top is hi(x) and index in bottom is lo(x).

Solution 3: Two-Level Bitvector with less Walking



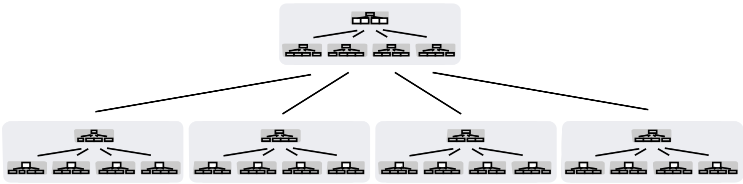
- Data structure. Solution 2 with min and max for each bottom structure.
- Predecessor(x):
  - If hi(x) in top and lo(x) ≥ min in bottom[lo(x)] walk left in bottom.
  - if hi(x) in top and lo(x) < min or hi(x) not in top walk left in top. Return max at first non-empty position in top.
- We either walk in bottom or top.
- Time. $O(u^{1/2})$
- Observation.
  - Query is walking left in one vector of size $u^{1/2}$ + O(1) extra work.
  - Why not walk using a predecessor data structure?

Solution 4: Two-Level Bitvector within Top and Bottom



- Data structure. Apply solution 3 to top and bottom structures of solution 3.
- Walking left in vector of size $u^{1/2}$ now takes $O((u^{1/2})^{1/2}) = O(u^{1/4})$ time.
- Each level adds O(1) extra work.
- Time. $O(u^{1/4})$
- Why not do this recursively?

Solution 5: van Emde Boas



- Data structure. Apply recursively until size of vectors is constant.
- Time. $T(u) = T(u^{1/2}) + O(1) = O(\log \log u)$
- Space. $O(u)$
  - Combined with perfect hashing we can reduce it to $O(n)$ [Mehlhorn and Näher 1990].

van Emde Boas

- Theorem. We can solve the static predecessor problem in
  - $O(n)$ space.
  - $O(\log \log u)$ time.
- Can also be made dynamic.

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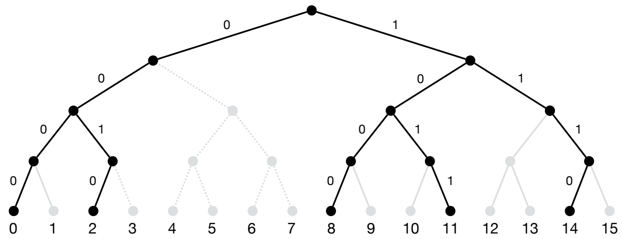
Tries

- Goal. Static predecessor with $O(n)$ space and $O(\log \log u)$ query time.
- Equivalent to van Emde Boas but different perspective. Simpler?
- Solution in 3 steps.
  - Trie. Slow and too much space.
  - X-fast trie. Fast but too much space.
  - Y-fast trie. Fast and little space.

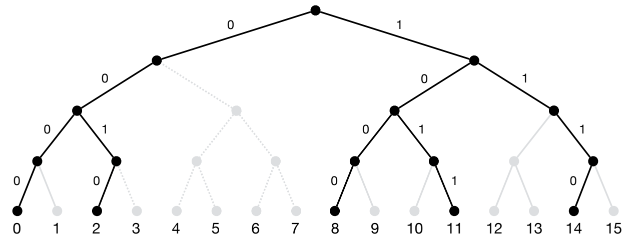
Tries



$S = \{0, 2, 8, 11, 14\} = \{0000_2, 0010_2, 1000_2, 1011_2, 1110_2\}$

- Trie. Tree T of prefixes of binary representation of keys in S.
  - Depth of T is log u
  - Number of nodes in T is O(n log u).

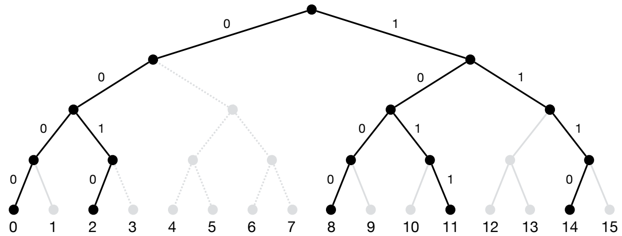
Solution 1: Top-down Traversal



$S = \{0, 2, 8, 11, 14\} = \{0000_2, 0010_2, 1000_2, 1011_2, 1110_2\}$

- Data structure.
  - T as binary tree with min and max for each node + keys ordered in a linked list.
- Predecessor(x): Top-down traversal to find the longest common prefix of x with T.
  - x branches of T to right ⟹ Predecessor(x) is max of sibling branch.
  - x branches of T to left ⟹ Successor(x) is min of sibling branch. Use linked list to get predecessor(x).
- Time. O(log u)
- Space. O(n log u)

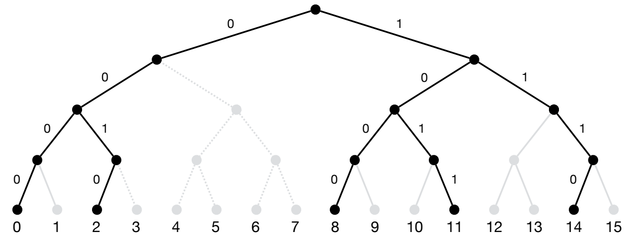
Solution 2: X-Fast Trie



$S = \{0, 2, 8, 11, 14\} = \{0000_2, 0010_2, 1000_2, 1011_2, 1110_2\}$

- Data structure.
  - For each level store a dictionary of prefixes of keys + solution 1.
  - Example. $d_1 = \{0,1\}$, $d_2 = \{00, 10, 11\}$, $d_3 = \{000, 001, 100, 101, 111\}$, $d_4 = S$
- Space. O(n log u)

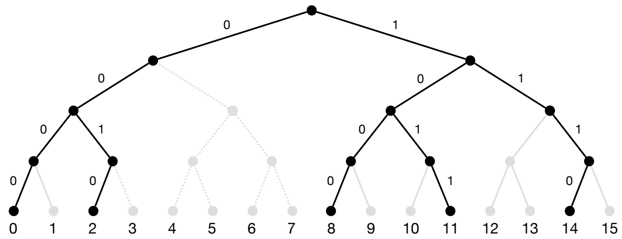
Solution 2: X-Fast Trie



$S = \{0, 2, 8, 11, 14\} = \{0000_2, 0010_2, 1000_2, 1011_2, 1110_2\}$

- Predecessor(x): Binary search over levels to find longest matching prefix with x.
- Example. Predecessor($9 = 1001_2$):
  - 10_2 in d_2 exists ⟹ continue in bottom 1/2 of tree.
  - 100_2 in d_3 exists ⟹ continue in bottom 1/4 of tree.
  - 1001_2 in d_4 does not exist ⟹ 100_2 is longest prefix.
- Time. O(log log u)

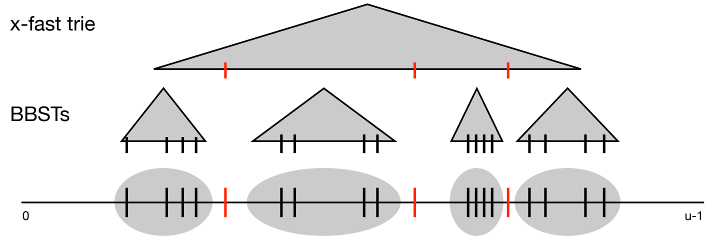
Solution 2: X-Fast Trie



S = {0, 2, 8, 11, 14} = $\{0000_2, 0010_2, 1000_2, 1011_2, 1110_2\}$

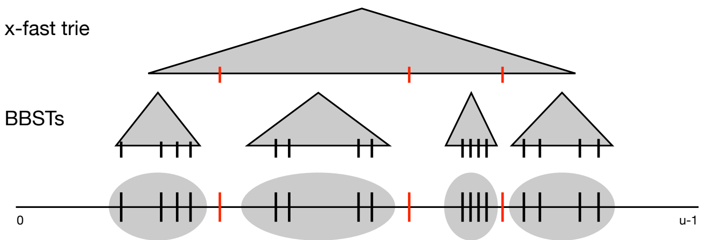
- Theorem. We can solve the static predecessor problem in
  - O(log log u) time
  - O(n log u) space.
- How do we get linear space?

Solution 3: Y-Fast Trie



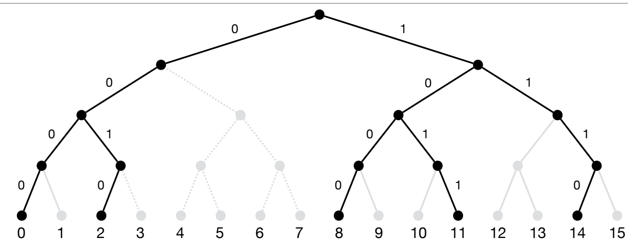
- Bucketing.
  - Partition S into O(n / log u) groups of log u consecutive keys.
  - Compute S' = set of split keys between groups. |S'| = O(n/log u)
- Data structure. x-fast trie over S' + balanced binary search trees for each group.
- Space.
  - x-fast trie: O(|S'| log u) = O(n/ log u · log u) = O(n).
  - Balanced binary search trees: O(n).
  - ⟹ O(n) in total.

Solution 3: Y-Fast Trie



- Predecessor(x):
  - Compute s = predecessor(x) in x-fast trie.
  - Compute predecessor(x) in BBST to the left or right of s.
- Time.
  - x-fast trie: O(log log u)
  - balanced binary search tree: O(log (group size)) = O(log log u).
  - ⟹ O(log log u) in total.

Solution 3: Y-Fast Trie



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- Theorem. We can solve the static predecessor problem in
  - O(log log u) time
  - O(n) space.

Y-Fast Tries

- Theorem. We can solve the static predecessor problem in
  - O(n) space.
  - O(log log u) time.
- What about updates?
- Theorem. We can solve the dynamic predecessor problem in
  - O(n) space
  - O(log log u) expected time for predecessor and updates.

From dynamic hashing

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