Approximation Algorithms

Approximation algorithms

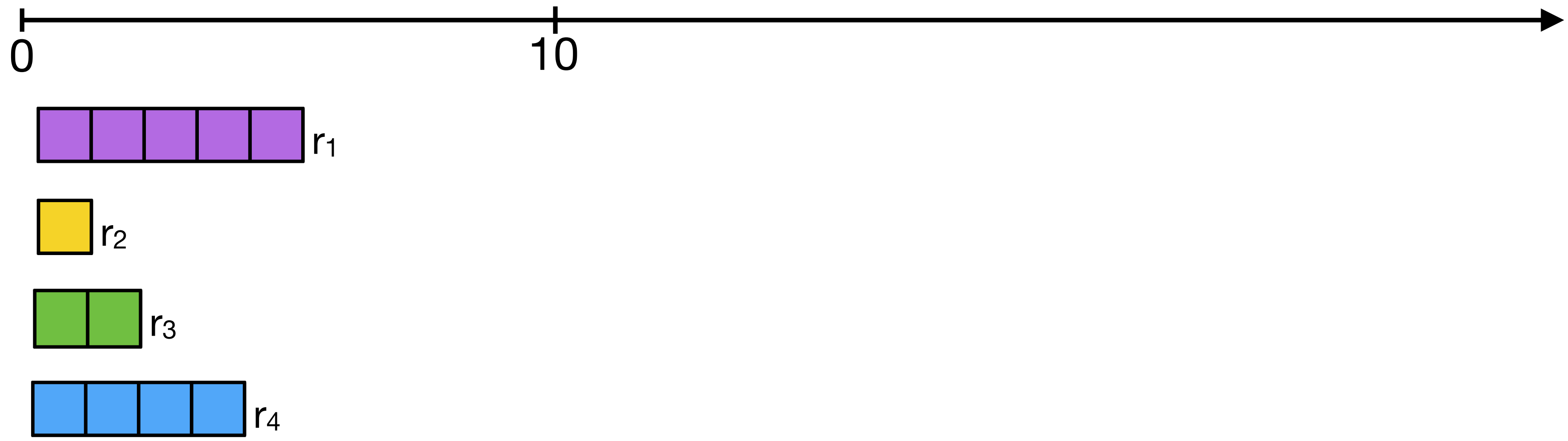
- Fast. Cheap. Reliable. Choose two.
- NP-hard problems: choose 2 of
  - optimal
  - polynomial time
  - all instances

- **Approximation algorithms.** Trade-off between time and quality.

- Let A(I) denote the value returned by algorithm A on instance I. Algorithm A is an *α-approximation algorithm* if for any instance I of the optimization problem:
  - A runs in polynomial time
  - A returns a valid solution
  - $A(I) \leq \alpha \cdot OPT$, where $\alpha \geq 1$, for minimization problems
  - $A(I) \geq \alpha \cdot OPT$, where $\alpha \leq 1$, for maximization problems

Scheduling

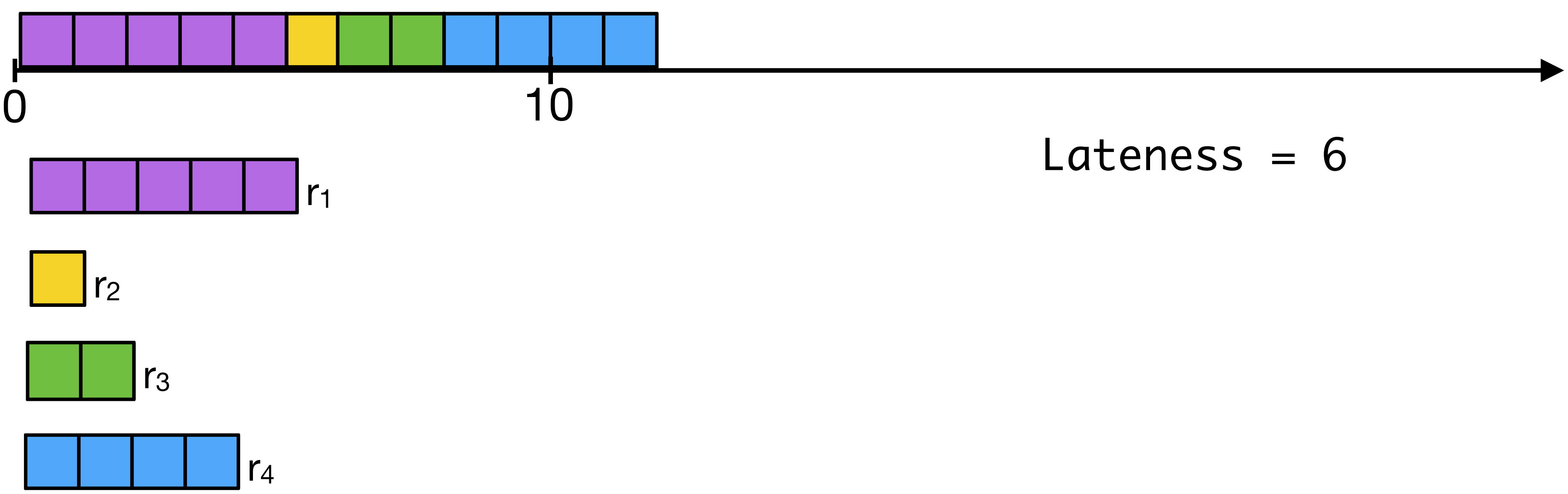
Scheduling jobs on a single machine



- n jobs
- Each job j has: processing time p_j, release date r_j, due date d_j.
- Once a job has begun processing it must be completed.
- Schedule starts at time 0.
- Lateness of job j completed at time C_j: $L_j = C_j - d_j$.
- Goal. Schedule all jobs so as to *minimize the maximum lateness*:

$$\text{minimize } L_{max} = \max_{i=1\ldots n} L_j$$

Scheduling jobs on a single machine



- n jobs
- Each job j has: processing time p_j, release date r_j, due date d_j.
- Once a job has begun processing it must be completed.
- Schedule starts at time 0.
- Lateness of job j completed at time C_j: $L_j = C_j - d_j$.
- Goal. Schedule all jobs so as to *minimize the maximum lateness*:

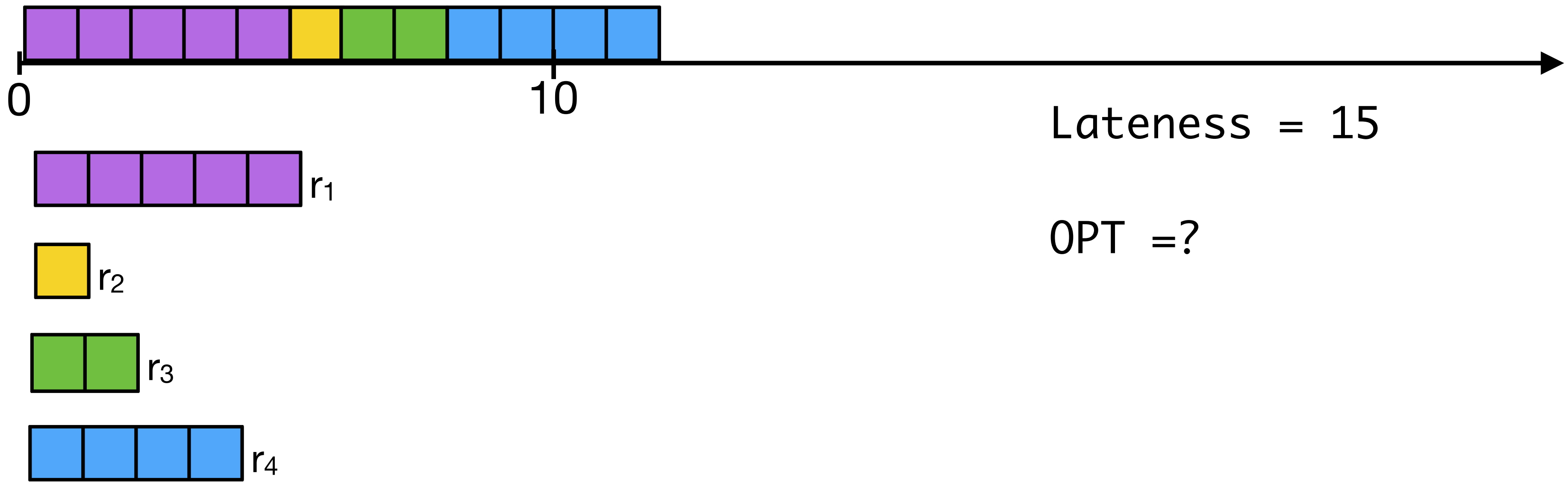
$$\text{minimize } L_{max} = \max_{i=1\ldots n} L_j$$

Scheduling jobs on a single machine

- NP-hard even to decide if all jobs can be completed by their due date.
- **Problem:** Assume optimal value is 0 then
  - α-approximation algorithm must find a solution of value at most $\alpha \cdot 0 = 0$
  - no such algorithm exists unless P=NP.
  - **Solution:** Assume all due dates are negative (optimal value always positive).

Earliest Due Date Rule

- Earliest due date rule (EDD). When machine idle: start processing an available job with earliest due date.

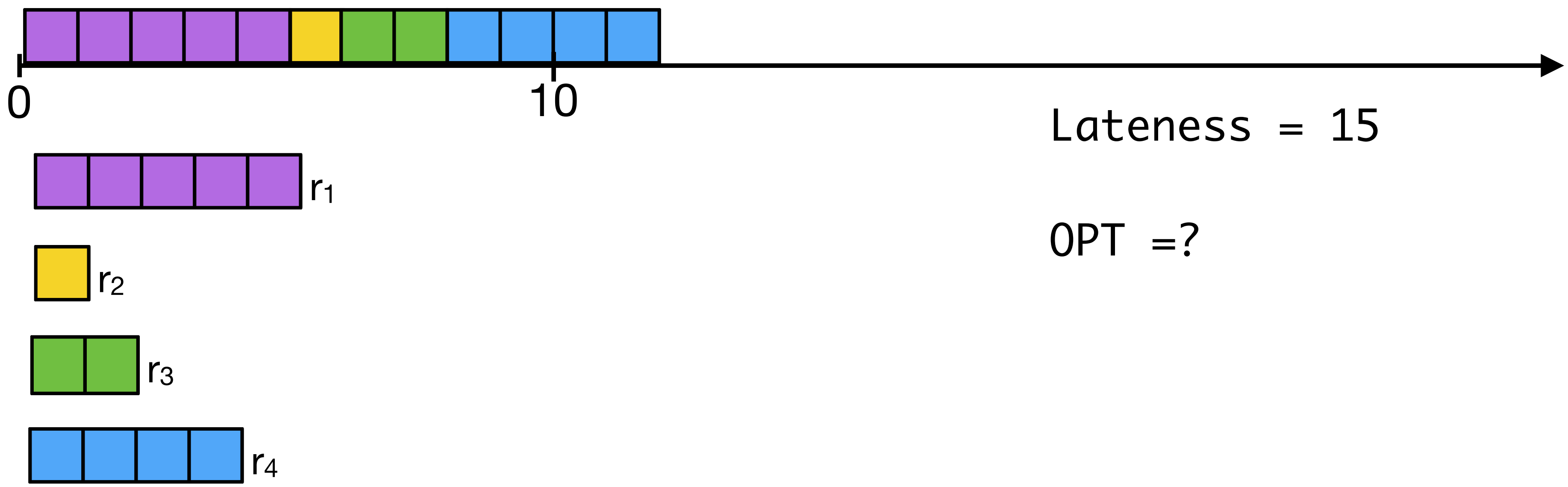


Lateness = 15

OPT =?

Earliest Due Date Rule

- **Earliest due date rule (EDD).** When machine idle: start processing an available job with earliest due date.



Lateness = 15

OPT =?

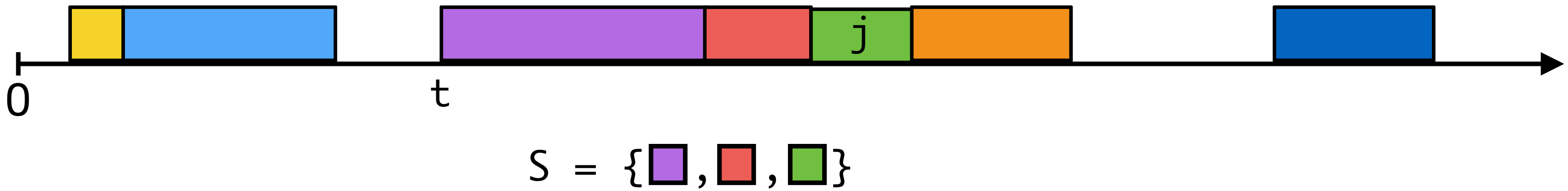
- EDD is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

Lower bound

- Let S be a subset of jobs
  - $r(S) = \min_{j \in S} r_j$
  - $p(S) = \sum_{\in S} p_j$
  - $d(S) = \max_{j \in S} d_j$
  - L^* optimal value

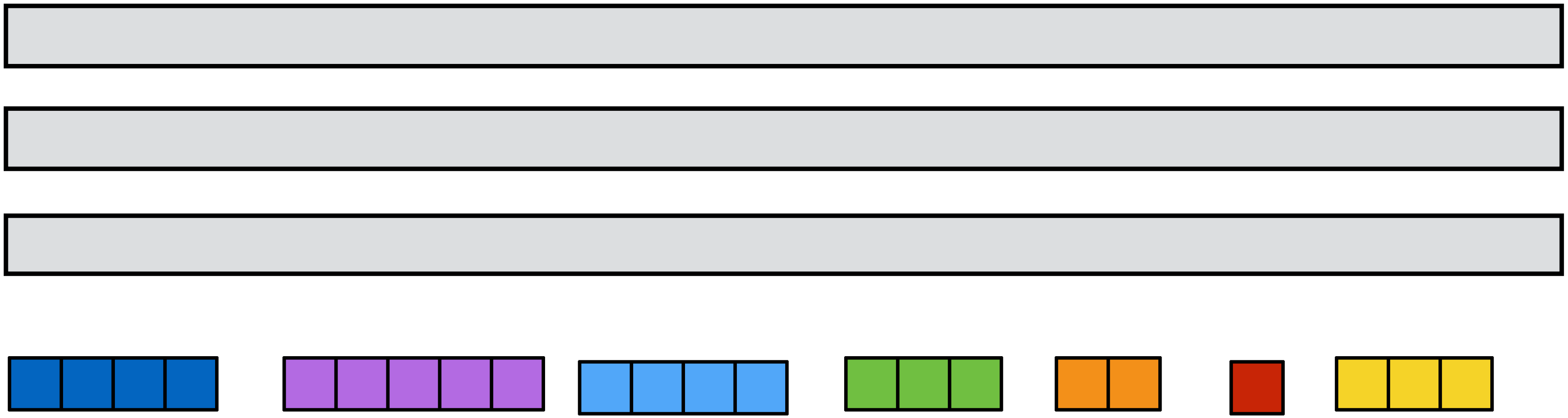
- **Claim.** For any subset S of jobs: $L^* \geq r(S) + p(S) - d(S)$.
- **Proof.**
  - Look at optimal schedule restricted to S.
  - No job can be processed before r(S).
  - Needed processing time p(S).
  - Latest job i to be processed cannot complete earlier than r(S) + p(S).
  - $d_i \leq d(S)$ => lateness of i at least r(S) + p(S) - d(S).
  - $L^* \geq L_i$.

EDD: Approximation factor



- j: job with maximum lateness ($L_{max} = L_j = C_j - d_j$).
- t: earliest time before C_j that machine idle (not idle in $[t, C_j)$).
- S: jobs processed in $[t, C_j)$.
- We have:
  - $r(S) = t$ and $p(S) = C_j - t$.
  - $C_j = p(S) + t = p(S) + r(S)$.
- Use Claim:
  - $L^* \geq r(S) + p(S) - d(S) \geq r(S) + p(S) = C_j$.
  - $L^* \geq r_j + p_j - d_j \geq - d_j$.
- $L_{max} = C_j - d_j \leq 2L^*$.

Scheduling on identical parallel machines



- n jobs to be scheduled on m identical machines.
- Each job has a processing time p_j.
- Once a job has begun processing it must be completed.
- Schedule starts at time 0.
- Completion time of job j = C_j.
- Goal. Schedule all jobs so as to *minimize the maximum completion time (makespan)*:

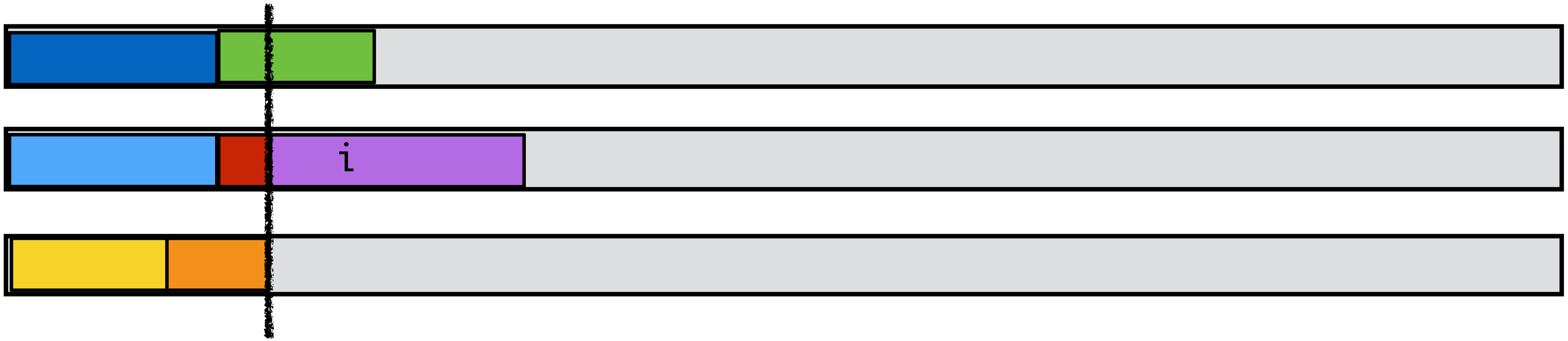
$$\text{minimize } C_{max} = \max_{i=1\ldots n} C_j$$

Local search

- Start with any schedule
- Consider job that finishes last:
  - If reassigning it to another machine can make it complete earlier, reassign it to the one that makes it finish earliest.
- Repeat until last finishing job cannot be transferred.

- The local search algorithm above is a 2-approximation algorithm:
  - polynomial time
  - valid solution ✓
  - factor 2

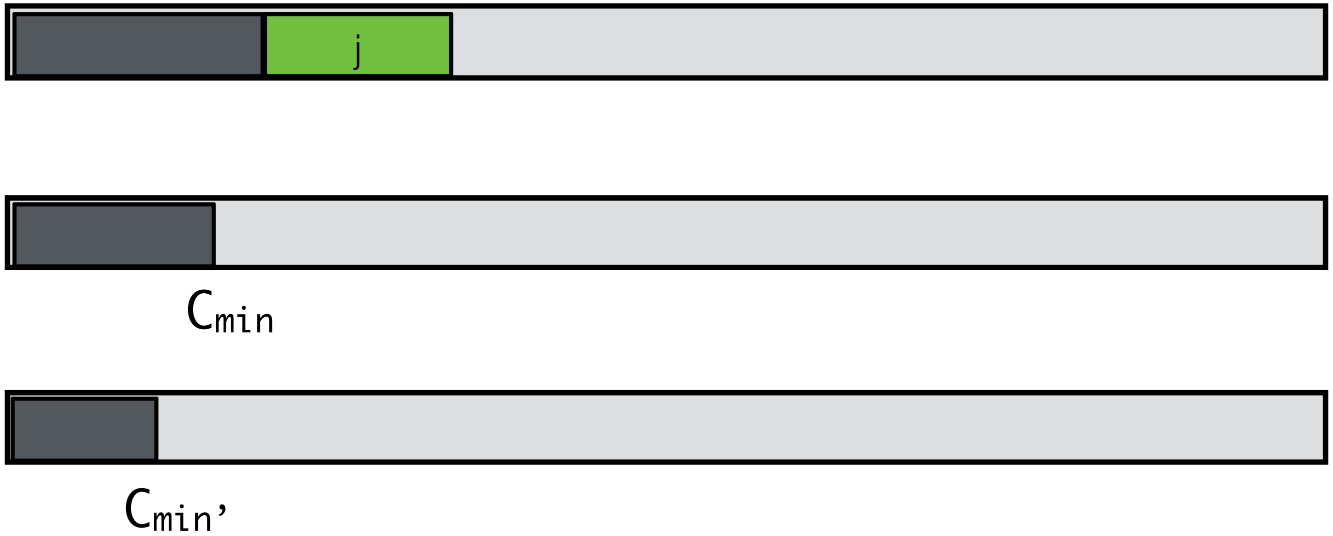
Approximation factor



- Each job must be processed: $C^* \geq \max_{i=1...n} p_j$
- There is a machine that is assigned at least average load: $C^* \geq \sum_{i=1...n} p_j/m$
- i: job finishes last.
- All other machines busy until start time s of i. ($s = C_i - p_i$)
- Partition schedule into before and after s.
- After $\leq C^*$.
- Before:
  - All machines busy => total amount of work = $m \cdot s$.
  - $m \cdot s \leq \sum_{i=1...n} p_i$ => $s \leq \sum_{i=1...n} p_i/m \leq C^*$.
- Length of schedule $\leq 2C^*$.

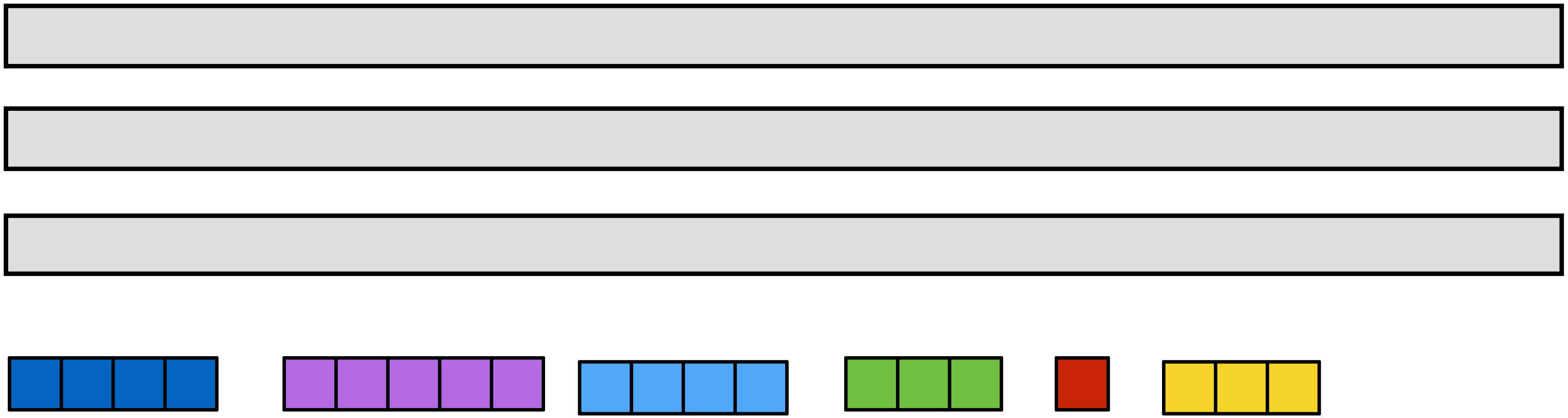
Running time

- Polynomial time. Does it terminate?
- Minimum completion time of machines C_{min} never decreases.
- Remains same => number of machines with minimum completion time decreases.
- No job transferred more than once:
  - Proof by contradiction. Assume j transferred twice.



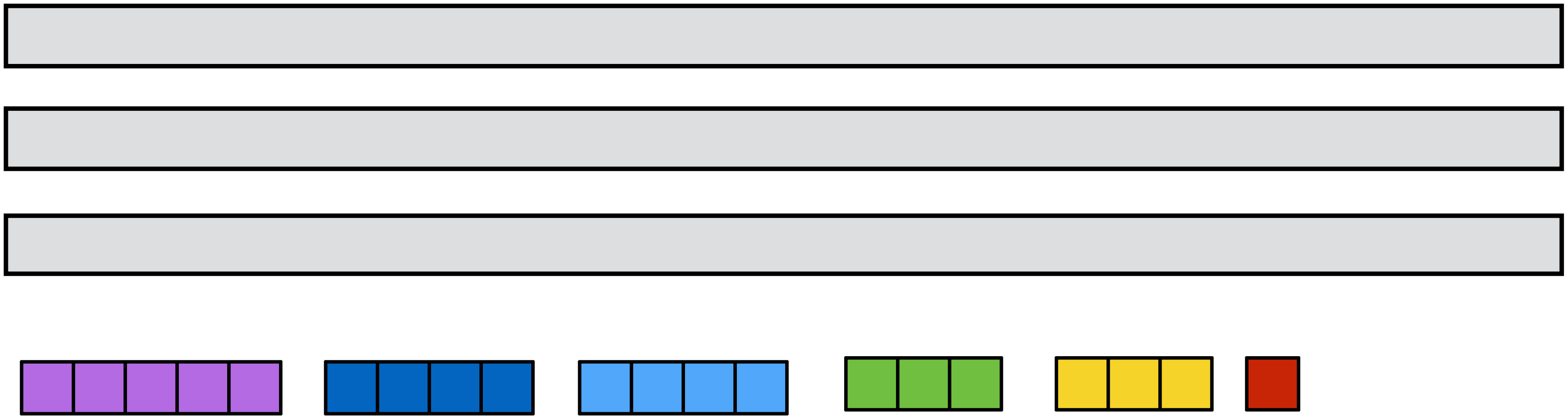
  - Then $C_{min} > C_{min'}$, but C_{min} does not decrease ↯

Longest processing time rule



- *Longest processing time rule (LPT).* Sort jobs in non-increasing order. Assign next job on list to machine as soon as it becomes idle.

Longest processing time rule



- *Longest processing time rule (LPT).* Sort jobs in non-increasing order. Assign next job on list to machine as soon as it becomes idle.
- LPT is a is a 4/3-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 4/3

Longest processing time rule

- **Longest processing time rule (LPT).** Sort jobs in non-increasing order. Assign next job on list to machine as soon as it becomes idle.
- Assume $p_1 \geq .... \geq p_n$.
- Assume wlog that smallest job finishes last.
- If $p_n \leq C^*/3$ then $C_{max} \leq 4/3\ C^*$.
- If $p_n > C^*/3$ then each machine can process at most 2 jobs.
- **Lemma.** *For any input where the processing time of each job is more than a third of the optimal makespan, LPT computes an optimal schedule.*

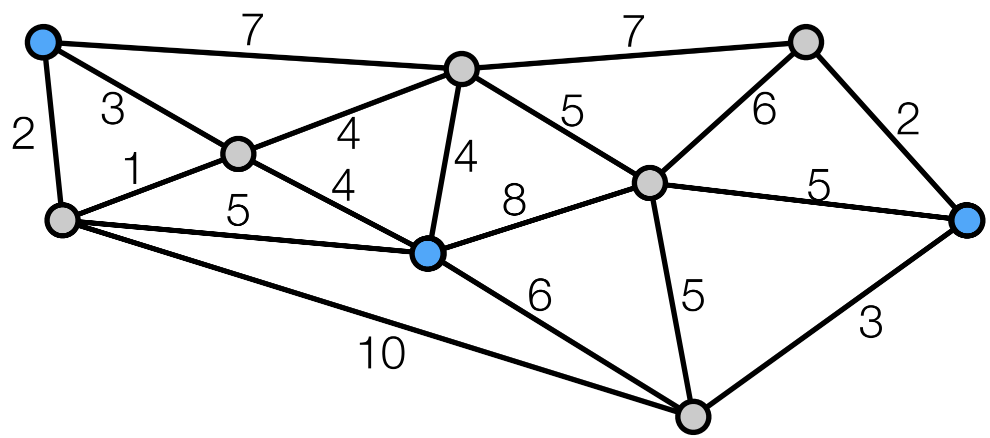
k-center

The k-center problem

- **Input.** An integer k and a complete, undirected graph G=(V,E), with distance d(i,j) between each pair of vertices $i,j \in V$.
- d is a metric:
  - $d(i,i) = 0$
  - $d(i,j) = d(j,i)$
  - $d(i,l) \leq d(i,j) + d(j,l)$
- **Goal.** Choose a set $S \subseteq V$, $|S| = k$, of k centers so as to minimize the maximum distance of a vertex to its closest center.

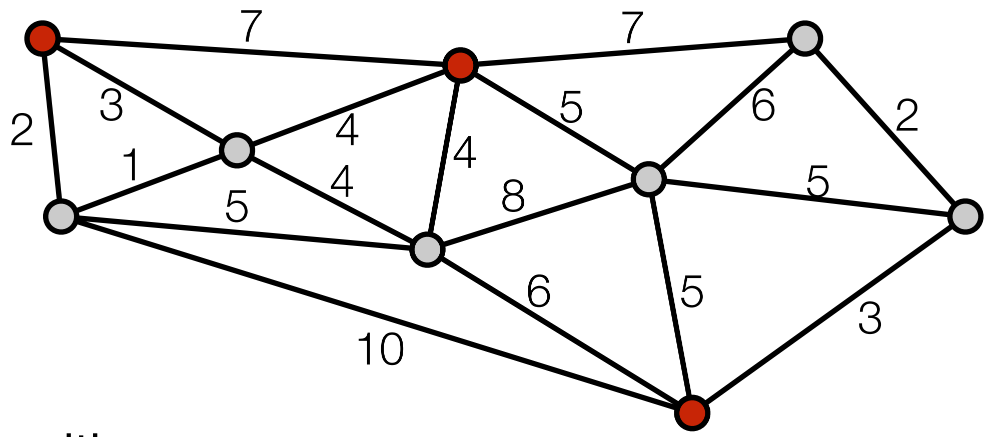
$$S = \text{argmin}_{S \subseteq V, |S|=k} \max_{i \in V} d(i,S)$$

- **Covering radius.** Maximum distance of a vertex to its closest center.



k-center: Greedy algorithm

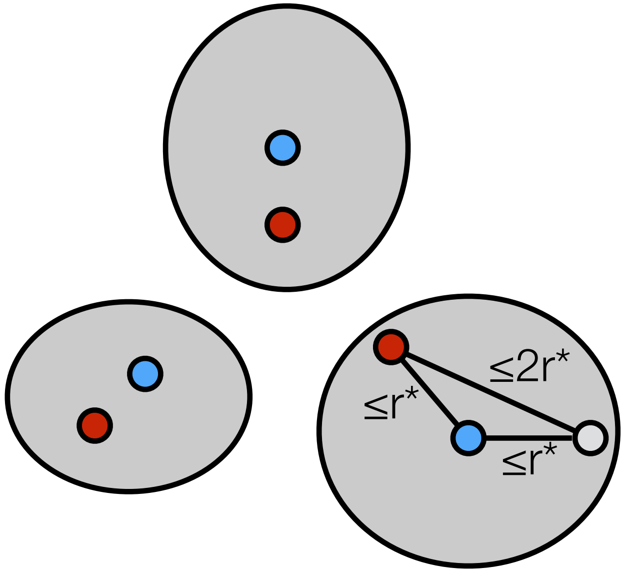
- Greedy algorithm.
  - Pick arbitrary i V.
  - Set S = {i}
  - while |S| < k do
    - Find vertex j farthest away from any cluster center in S
    - Add j to S



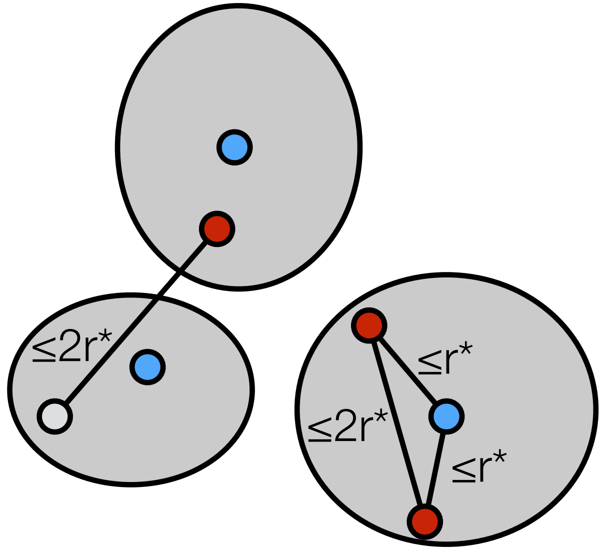
- Greedy is a 2-approximation algorithm:
  - polynomial time ✓
  - valid solution ✓
  - factor 2

k-center: analysis

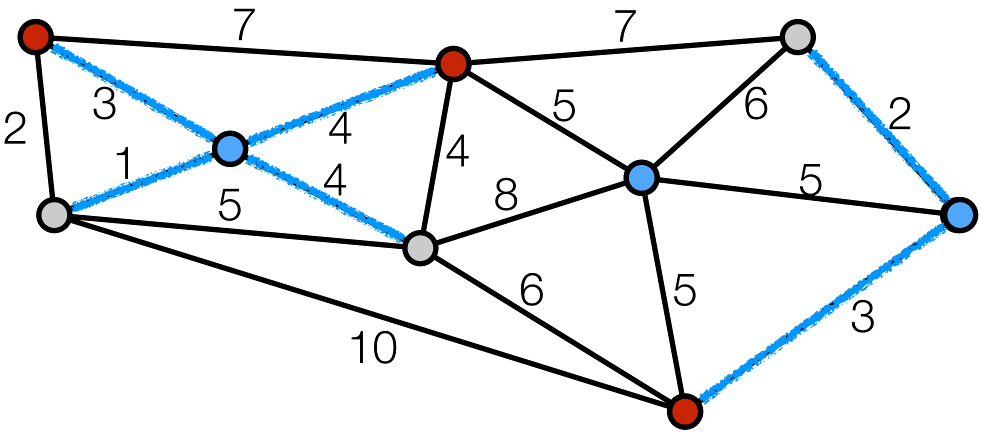
- r* optimal radius.
- Show all vertices within distance 2r* from a center.
- Consider optimal clusters. 2 cases.
  - Algorithm picked one center in each optimal cluster
    - distance from any vertex to its closest center ≤ 2r* (triangle inequality)



  - Some optimal cluster does not have a center.
    - Some cluster have more than one center.
    - distance between these two centers ≤ 2r*.
    - when second center in same cluster picked it was the vertex farthest away from any center.
    - distance from any vertex to its closest center at most 2r*.



k-center



k-center: Inapproximability

- There is no α-approximation algorithm for the k-center problem for $\alpha < 2$ unless P=NP.
- **Proof.** Reduction from dominating set.
- Dominating set: Given G=(V,E) and k. Is there a (dominating) set $S \subseteq V$ of size k, such that each vertex is either in S or adjacent to a vertex in S.
- Given instance of the dominating set problem construct instance of k-center problem:
    - Complete graph G' on V.
    - All edges from E has weight 1, all new edges have weight 2.
    - Radius in k-center instance 1 or 2.
    - G has an dominating set of size k <=> opt solution to the k-center problem has radius 1.
    - Use α-approximation algorithm A:
        - opt = 1 => A returns solution with radius at most $\alpha < 2$.
        - opt = 2 => A returns solution with radius 2.
        - Can use A to distinguish between the 2 cases.