02110

Inge Li Gørtz

Thank you to Kevin Wayne for inspiration to slides

Welcome

- Inge Li Gørtz.
- Reverse teaching and discussion of exercises:
  - 3 teaching assistants
  - 8.00-9.00 Group work (on exercises you could not solve at home)
  - 9.00-10.30 Discussions of your solutions in class
  - 10.30-12.00 Lecture
- Prepare and solve exercises at home.
- Use Piazza for questions during the week
- Weekly assignments (You have to pass 3 out of 9 + 2 implementation exercises in order to be able to attend the written exam).
- Prerequisites: 02105/02326 Algorithms and Data Structures I

Balanced Search Trees

2-3-4 trees
red-black trees

References: CLRS Chapter 13, Algorithms in Java (handout)

Balanced search trees

Dynamic sets

- Search
- Insert
- Delete
- Maximum
- Minimum
- Successor
- Predecessor

This lecture: 2-3-4 trees, red-black trees

Next time: Splay trees

Dynamic set implementations

Worst case running times

Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)

In worst case h=n.

In best case h= log n (fully balanced binary tree)

Today: How to keep the trees balanced.

2-3-4 trees

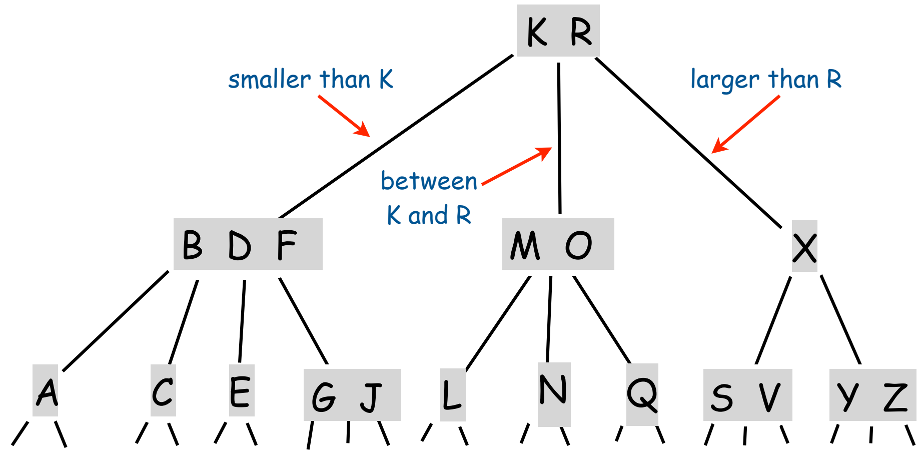
2-3-4 trees

2-3-4 trees. Allow nodes to have multiple keys.

Perfect balance. Every path from root to leaf has same length.

Allow 1, 2, or 3 keys per node

- 2-node: one key, 2 children
- 3-node: 2 keys, 3 children
- 4-node: 3 keys, 4 children

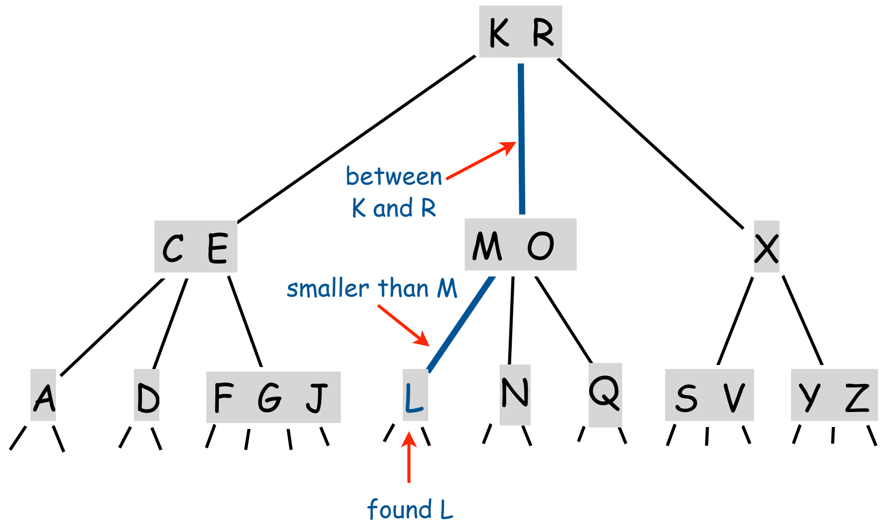


Searching in a 2-3-4 tree

Search.

- Compare search key against keys in node.
- Find interval containing search key
- Follow associated link (recursively)

Ex. Search for L

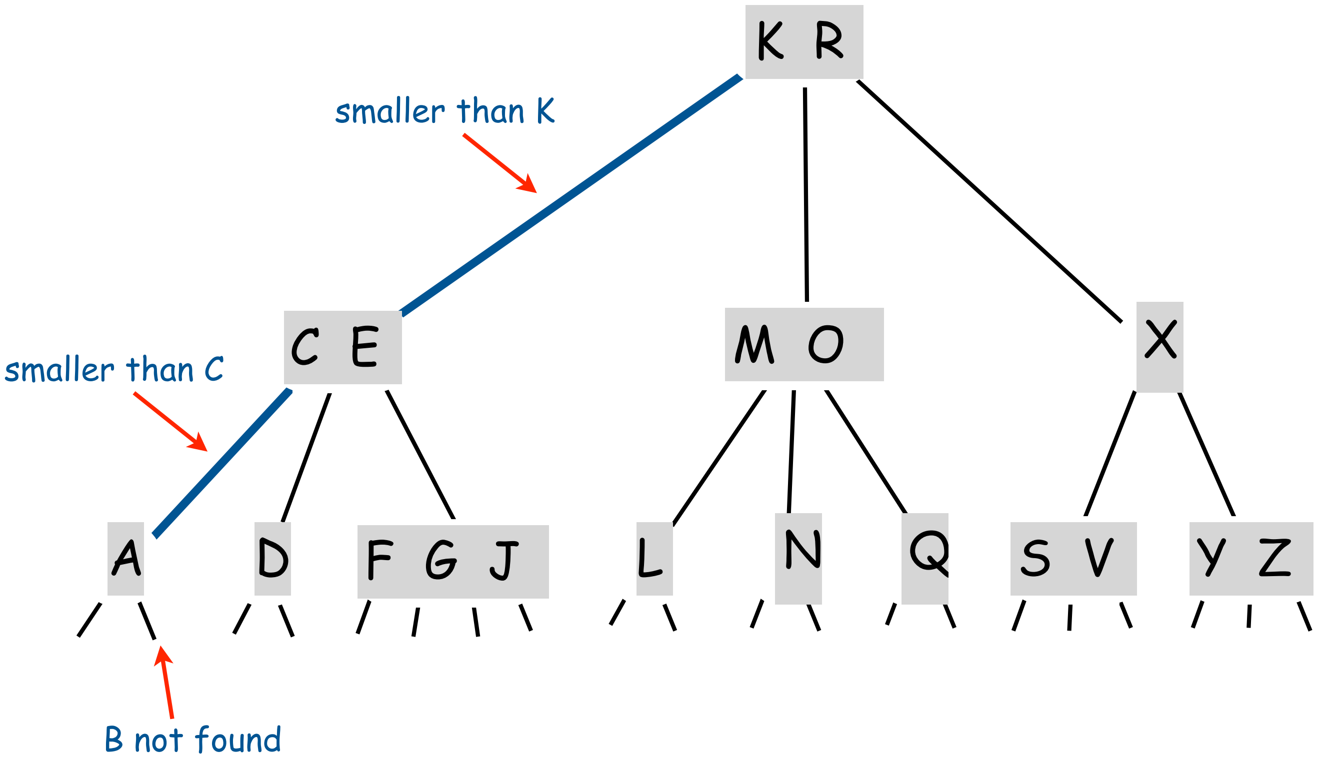


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.

Ex. Insert B

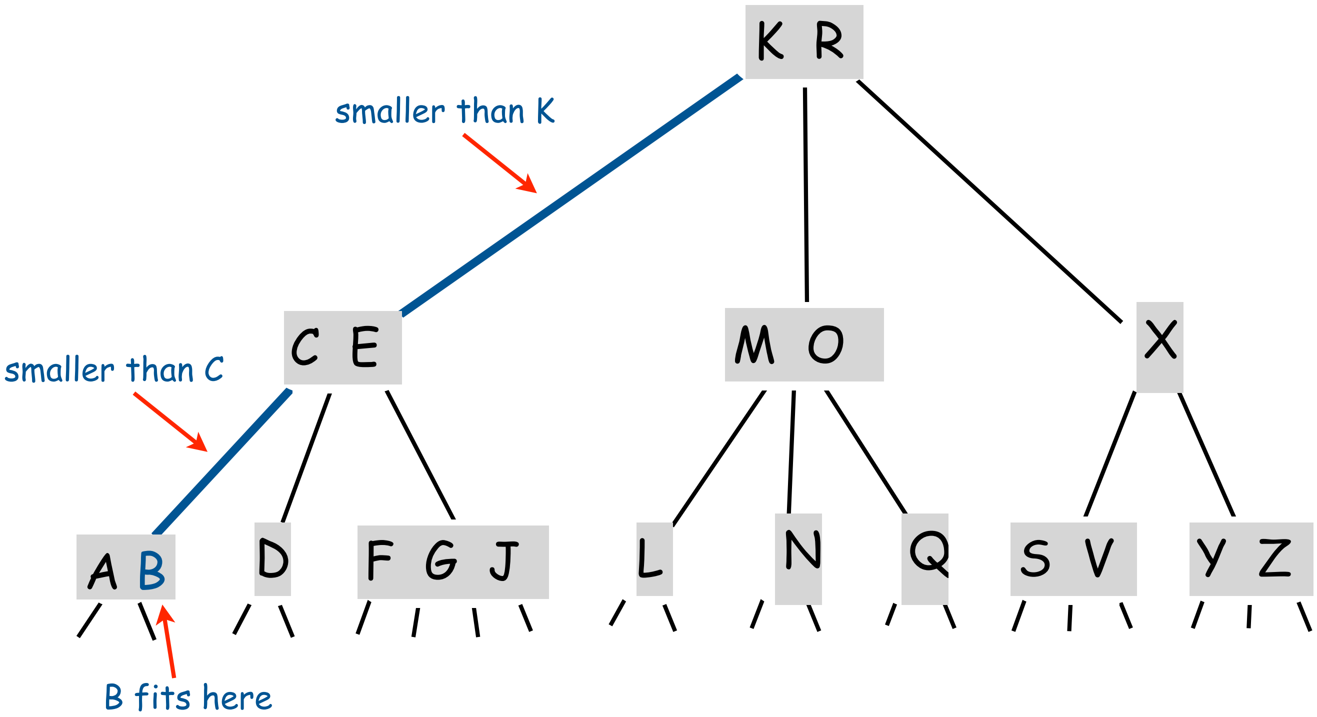


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node

Ex. Insert B

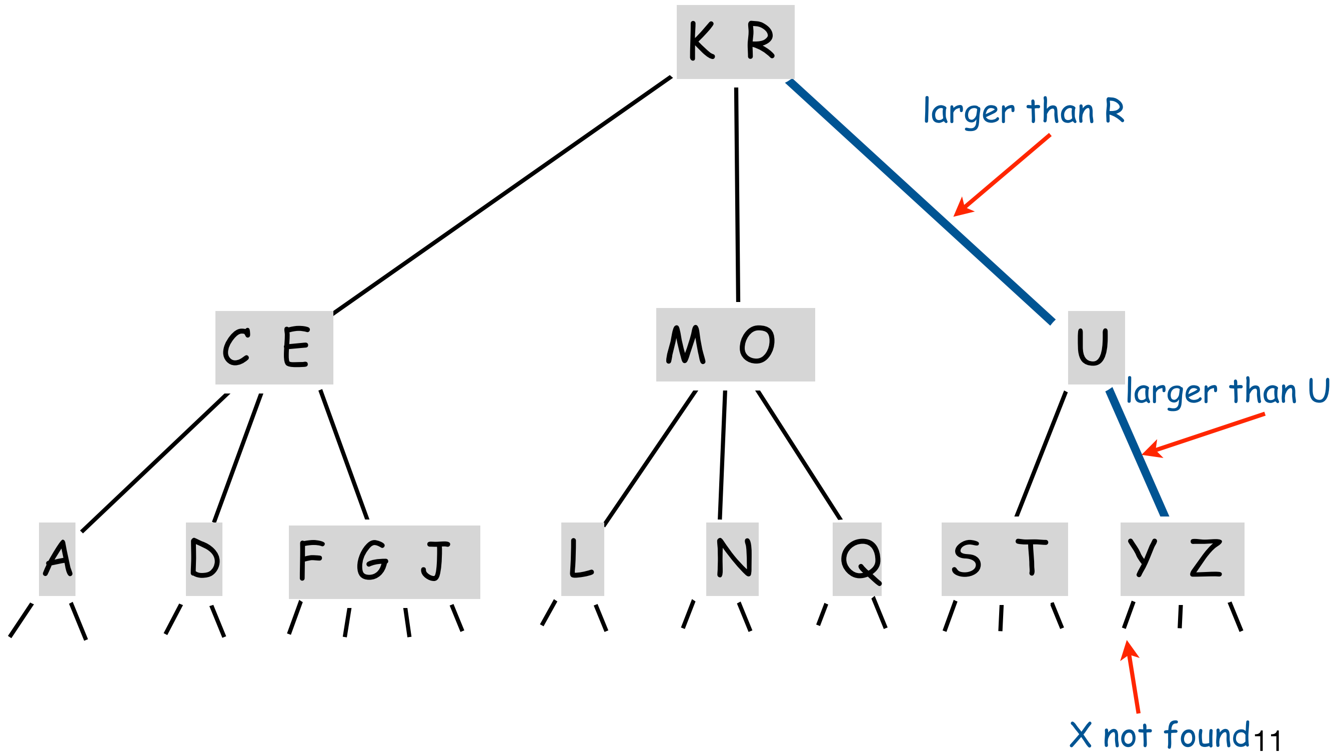


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node

Ex. Insert X

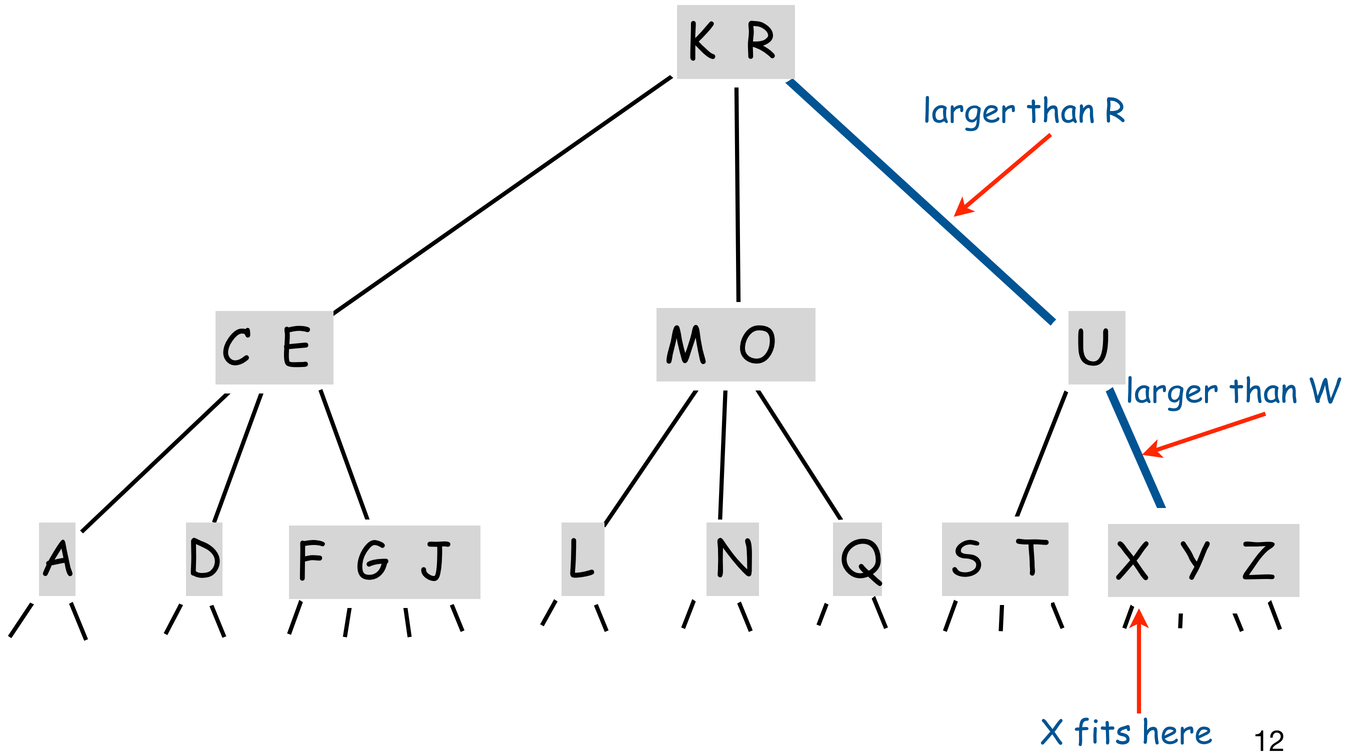


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node

Ex. Insert X

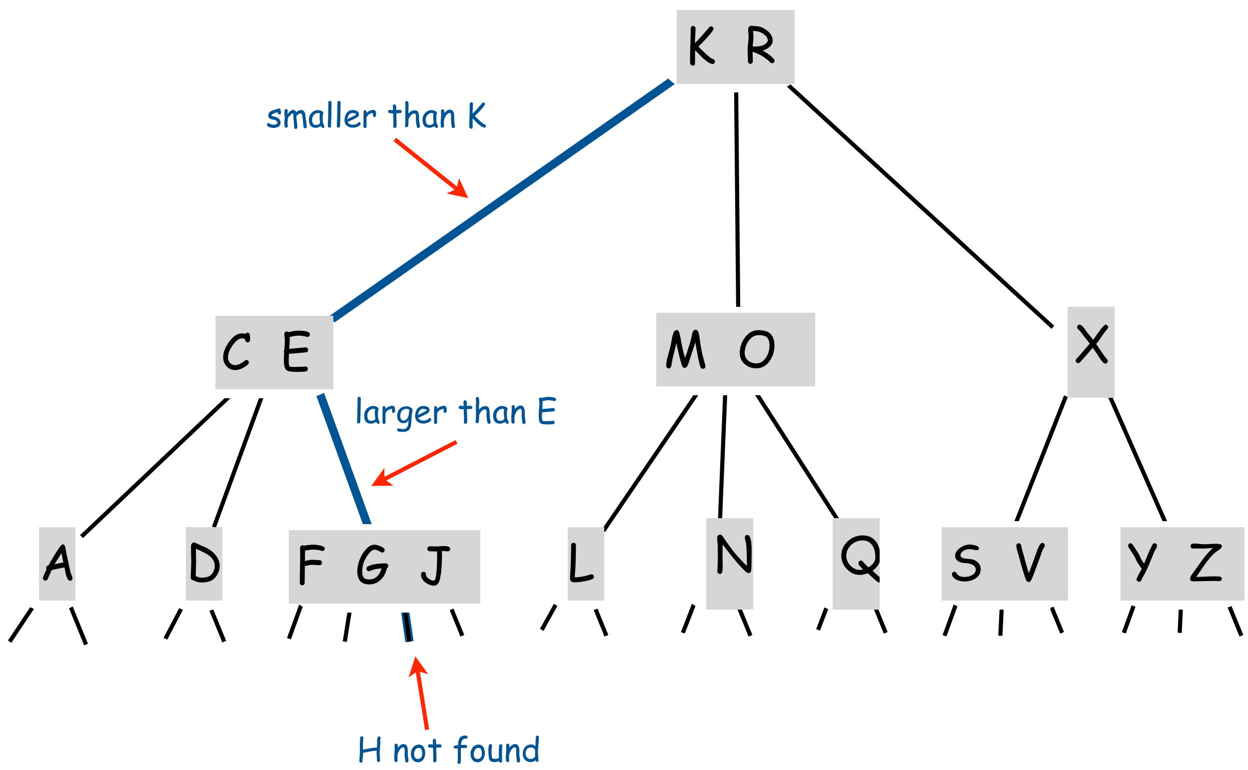


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node

Ex. Insert H

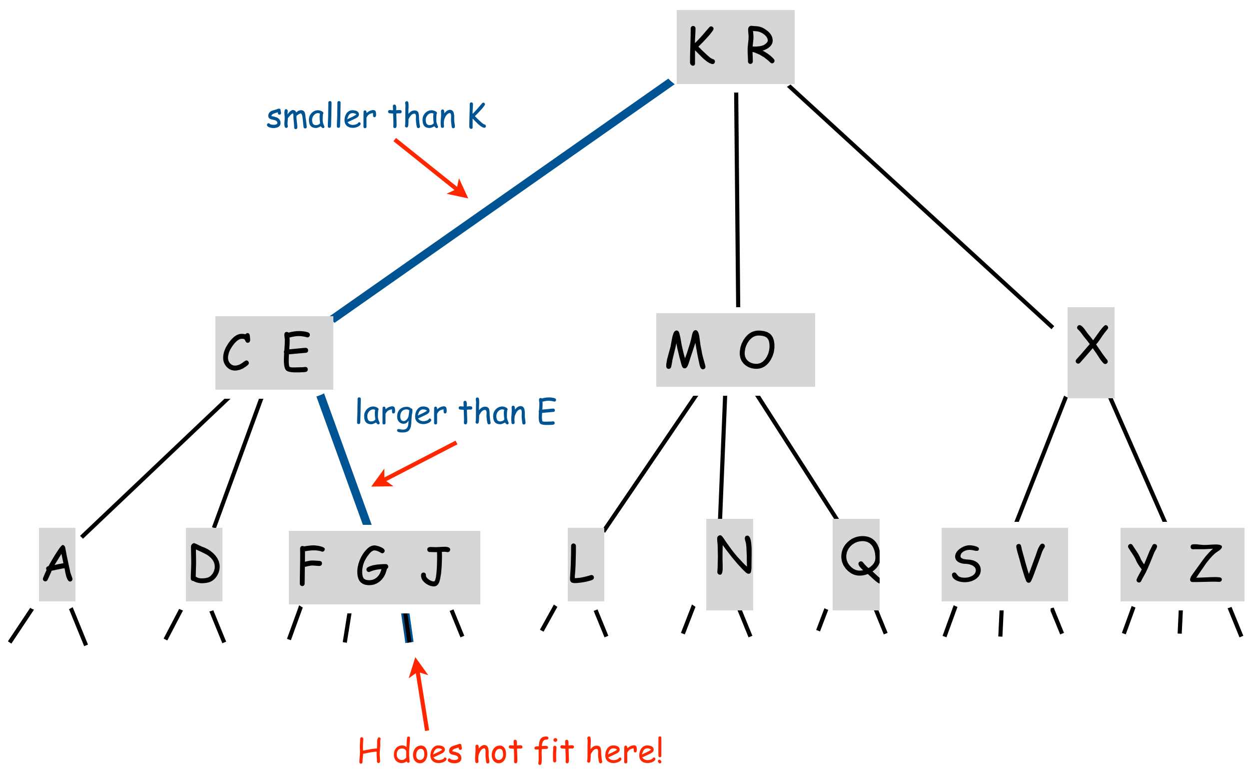


Insertion in a 2-3-4 tree

Insert.

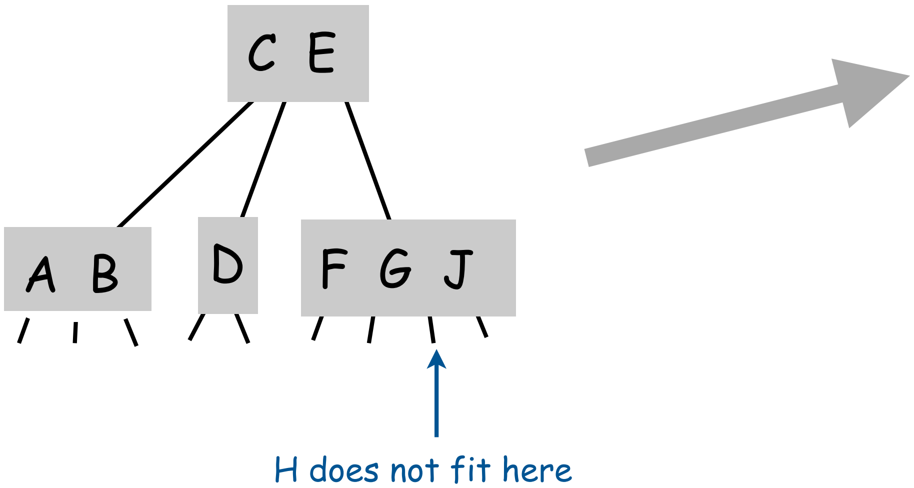
- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

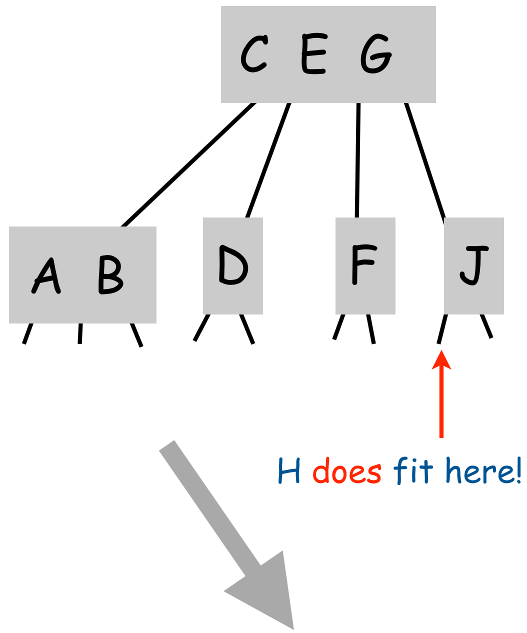
Ex. Insert H

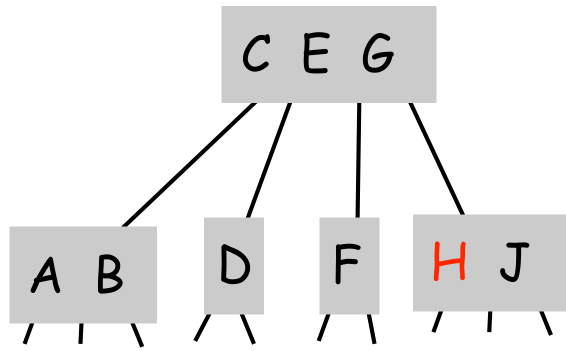


Splitting a 4-node in a 2-3-4 tree

Idea: split the 4-node to make room







Problem: Doesn’t work if parent is a 4-node

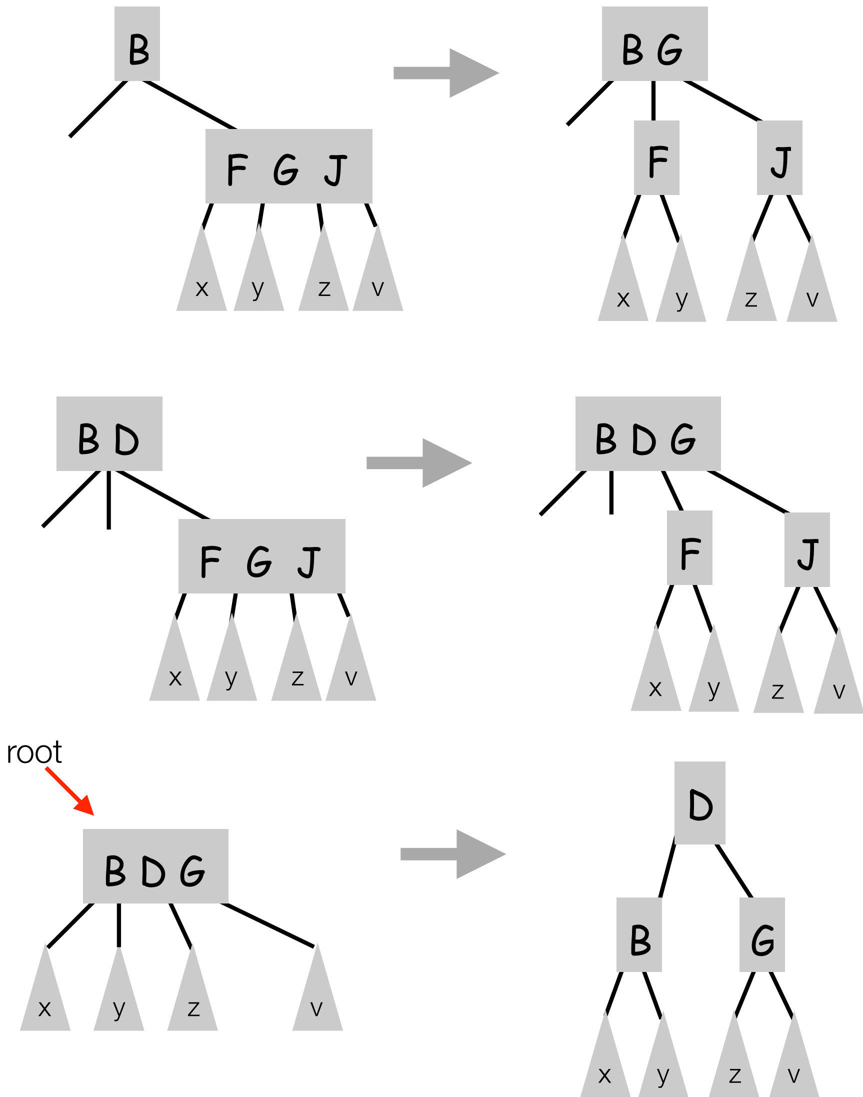
Solution 1: Split the parent (and continue splitting while necessary).

Solution 2: Split 4-nodes on the way down.

Splitting 4-nodes in a 2-3-4 tree

Idea: split 4-nodes on the way down the tree.

- Ensures last node is not a 4-node.
- Transformations to split 4-nodes:



Invariant. Current node is not a 4-node.

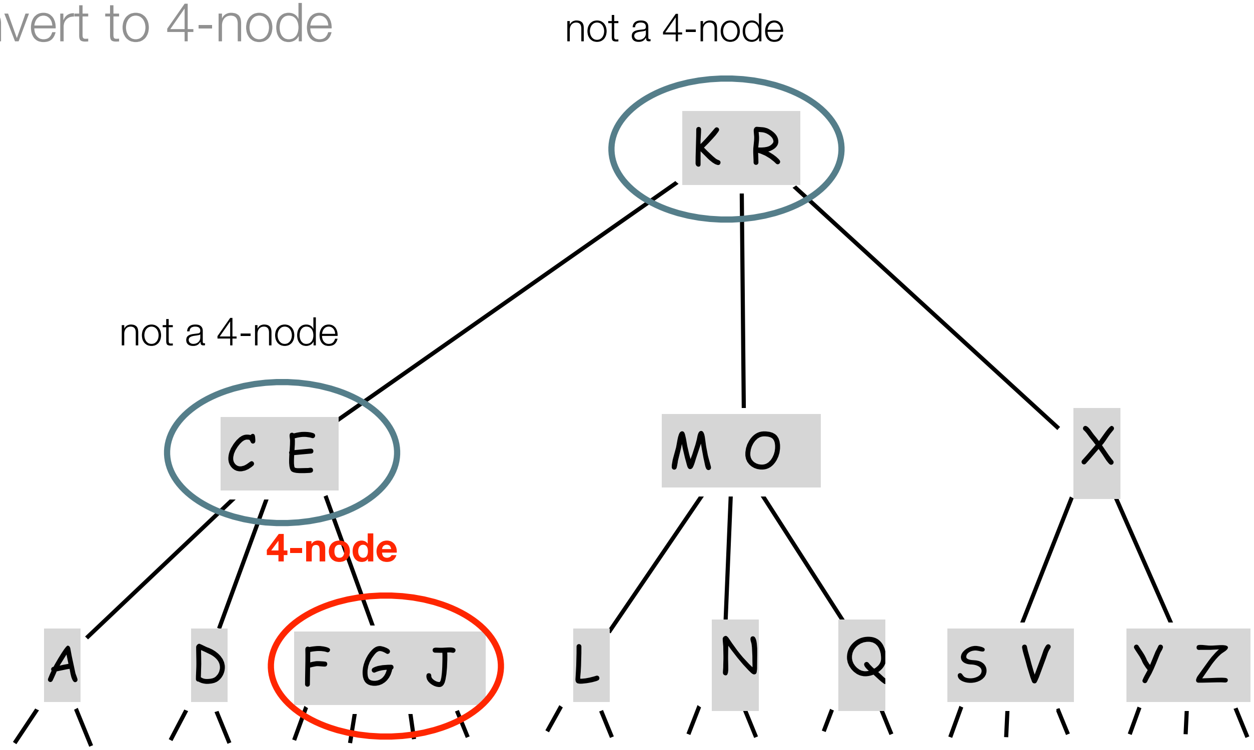
Consequence. Insertion at bottom is easy since it's not a 4-node.

Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

Ex. Insert H

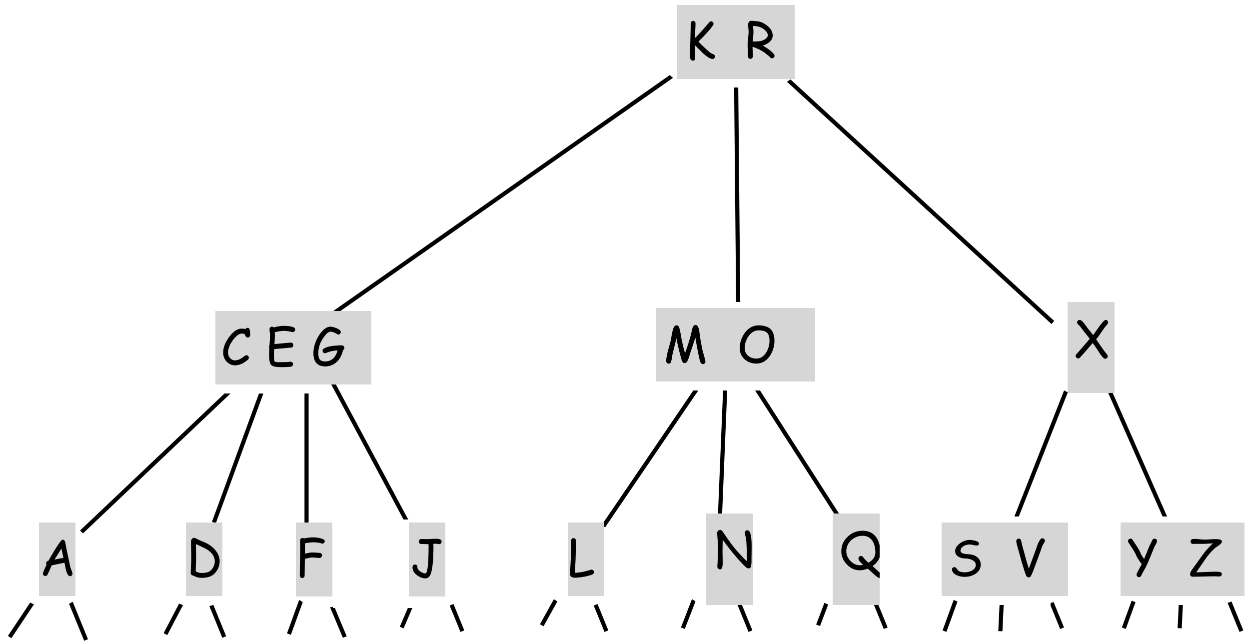


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

Ex. Insert H

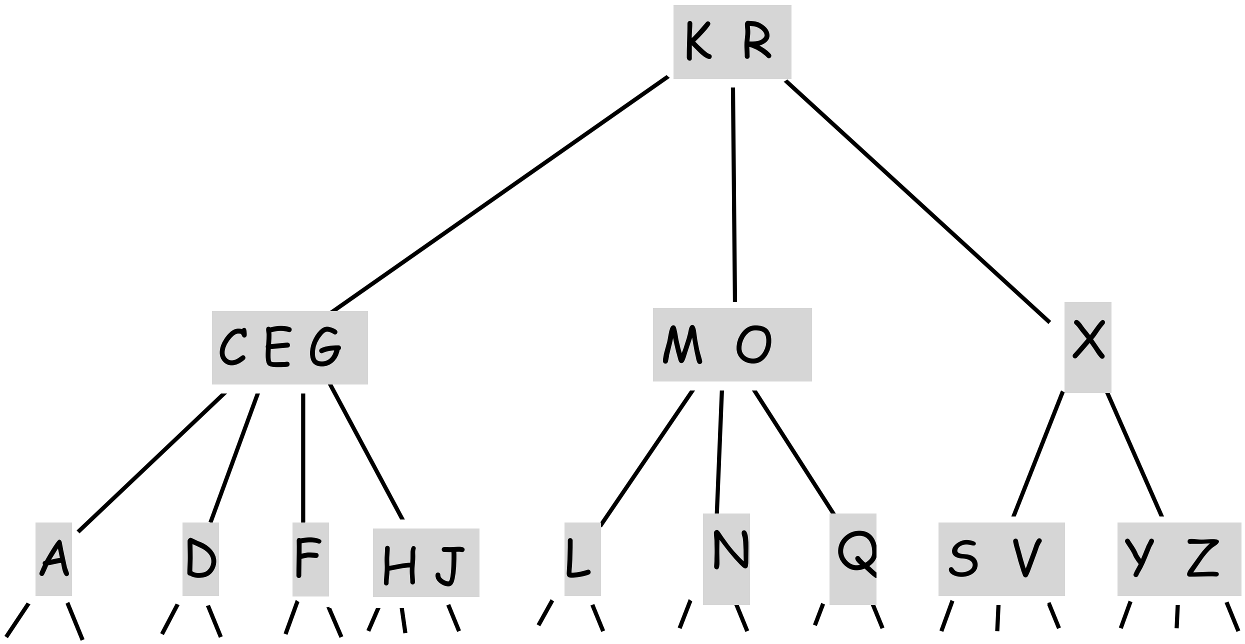


Insertion in a 2-3-4 tree

Insert.

- Search to bottom for key.
- 2-node at bottom: convert to 3-node
- 3-node at bottom: convert to 4-node
- 4-node at bottom: ??

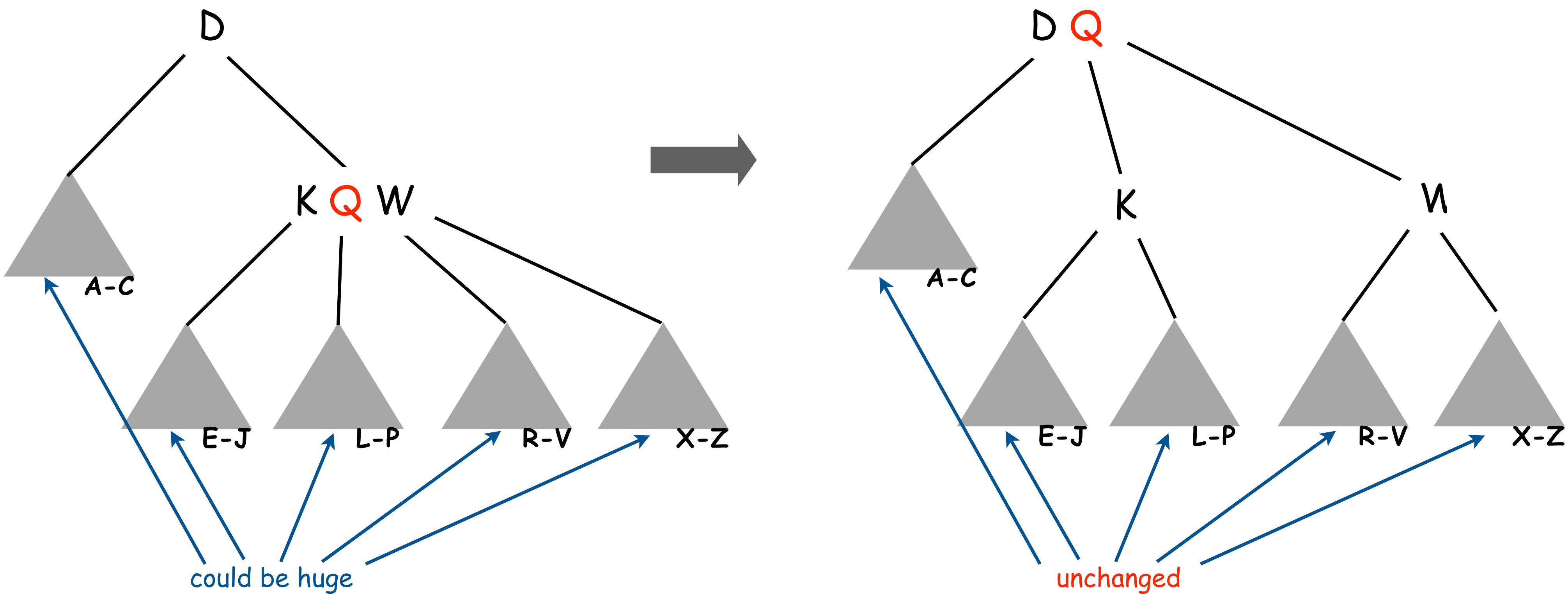
Ex. Insert H



Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree.

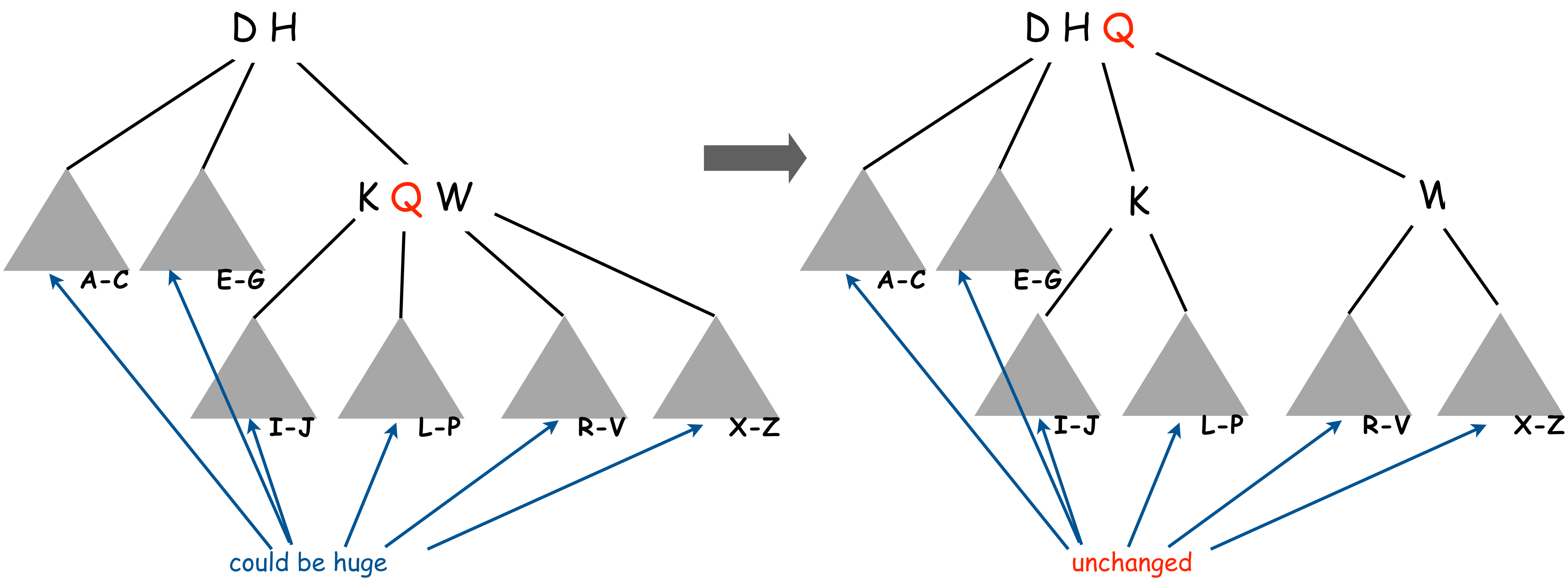
Ex. Splitting a 4-node attached to a 2-node



Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree

Ex. Splitting a 4-node attached to a 3-node



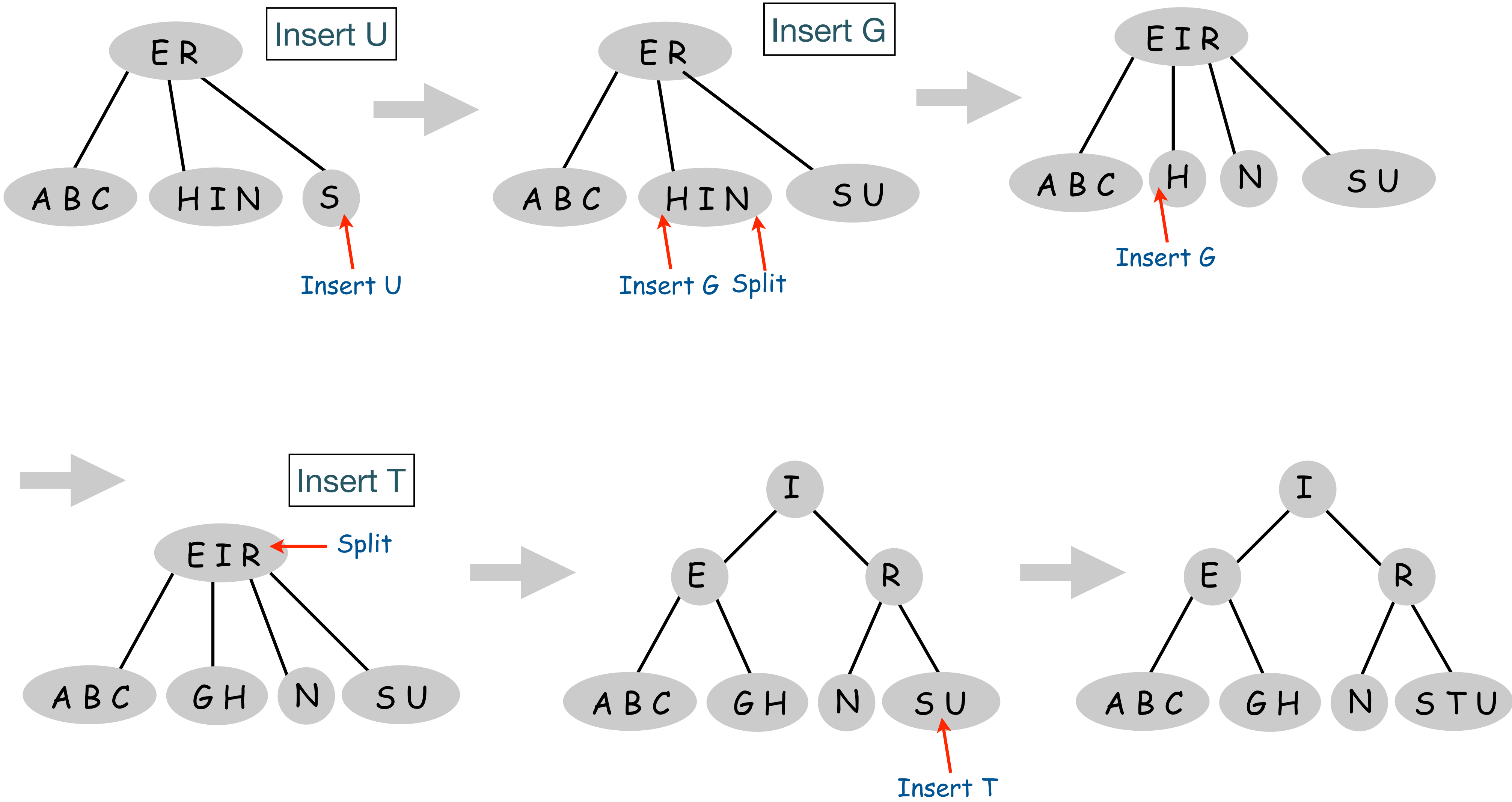
Splitting 4-nodes in a 2-3-4 tree

Local transformations that work anywhere in the tree.

Splitting a 4-node attached to a 4-node never happens when we split nodes on the way down the tree.

Invariant. Current node is not a 4-node.

Insertion 2-3-4 trees

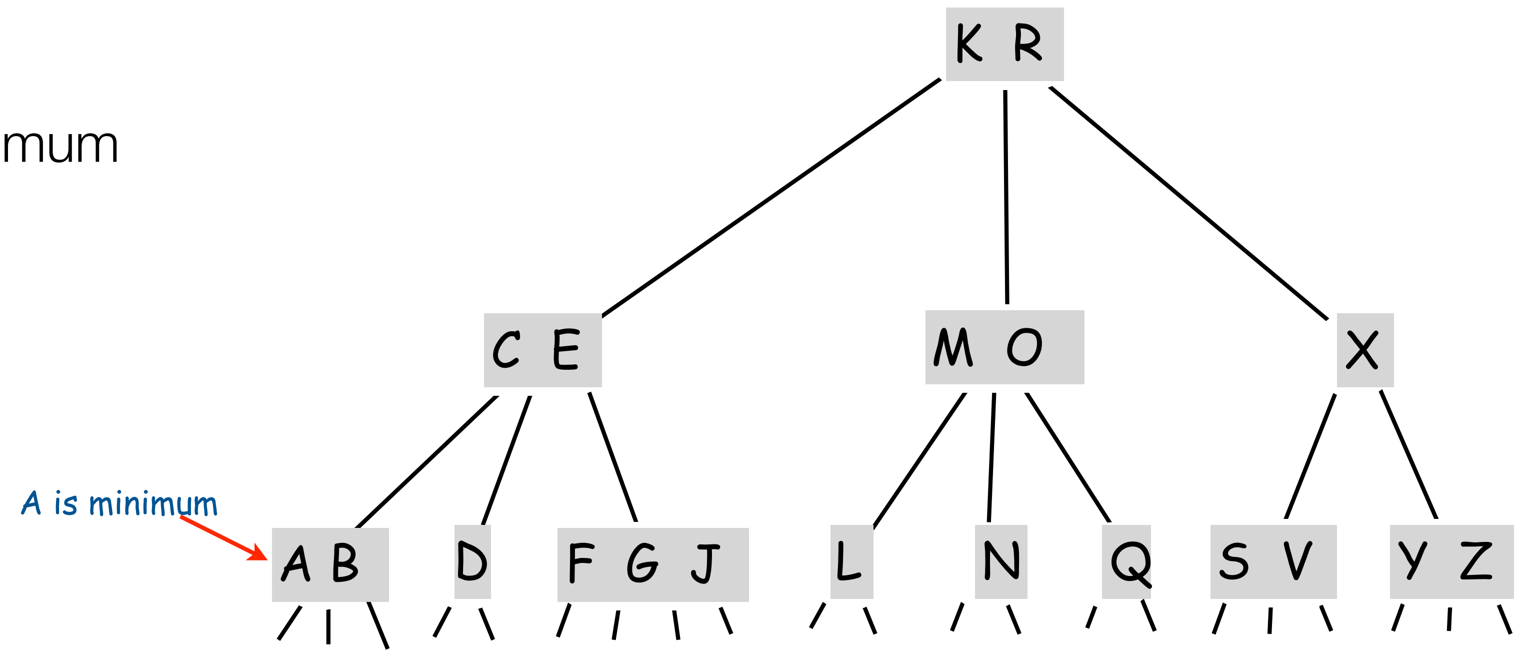


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key

Ex. Delete minimum

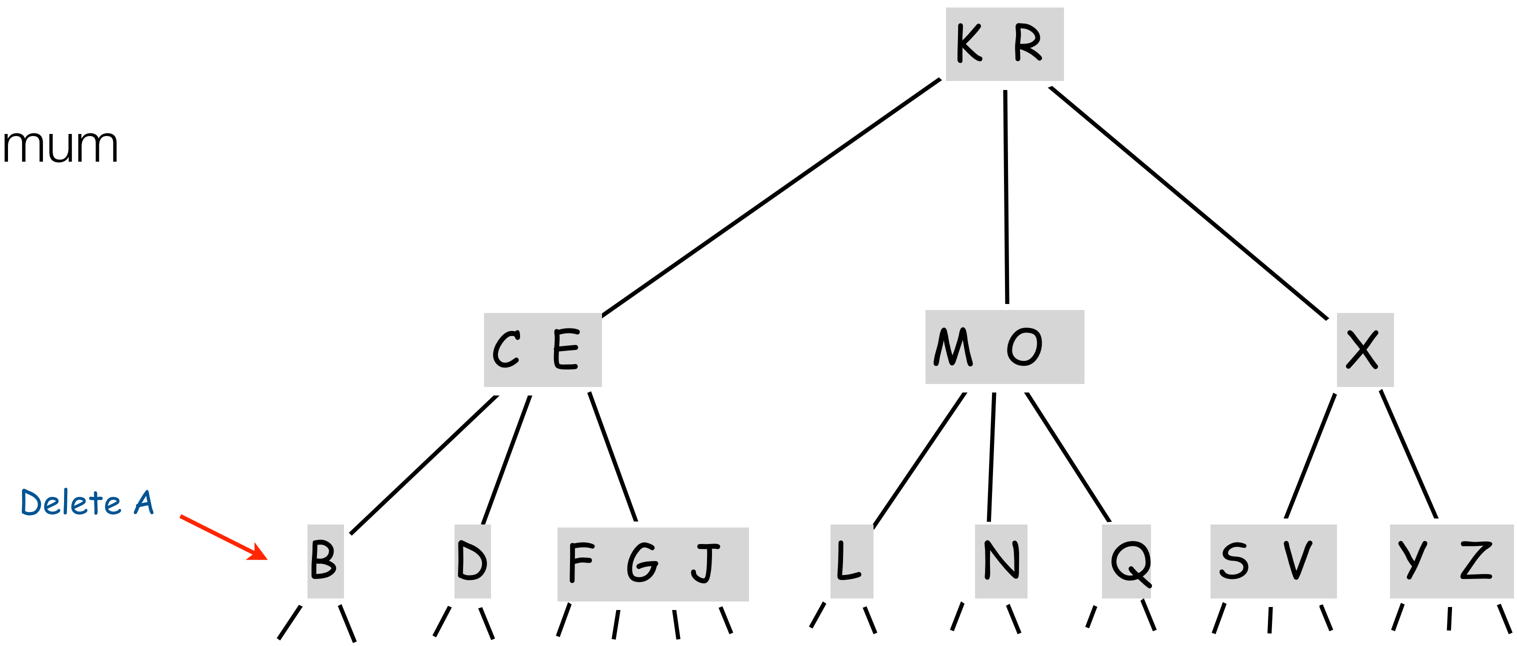


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key

Ex. Delete minimum

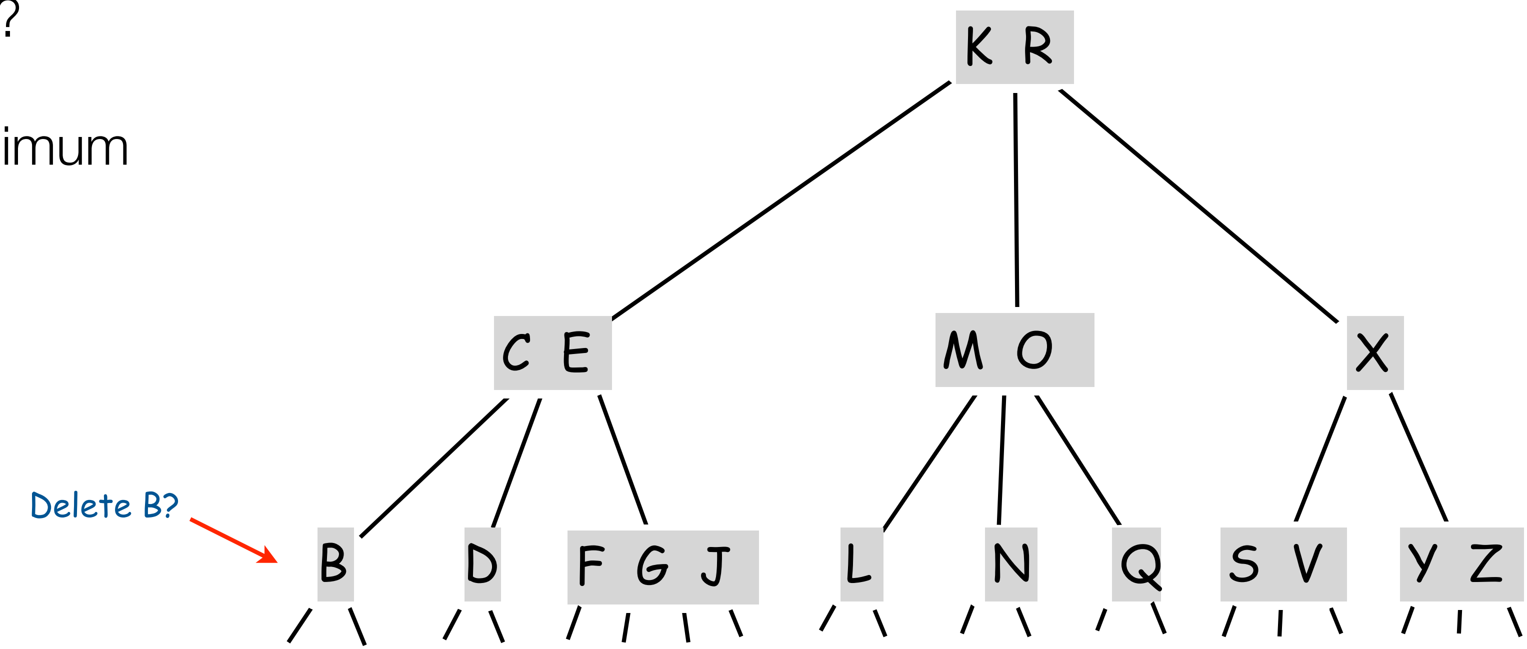


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node??

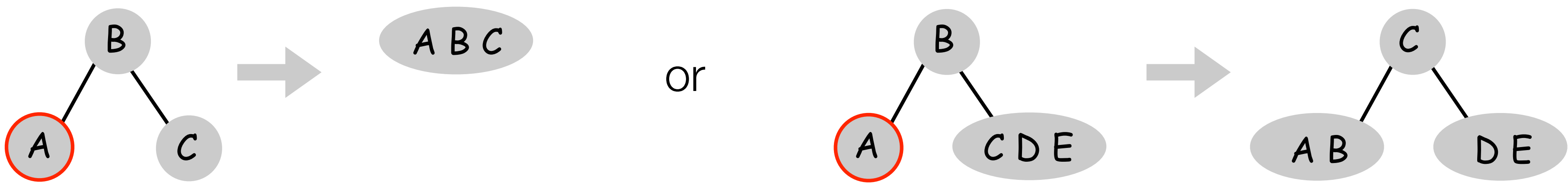
Ex. Delete minimum



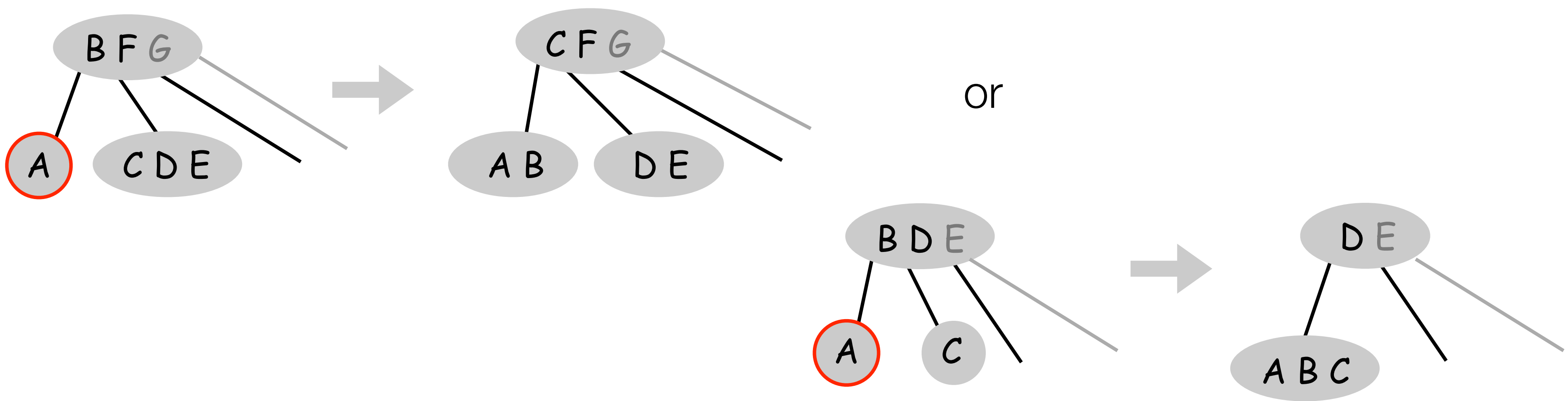
Deletions in 2-3-4 trees

Idea: On the way down maintain the invariant that current node is not a 2-node.

- Child of root and root is a 2-node:



- on the way down:

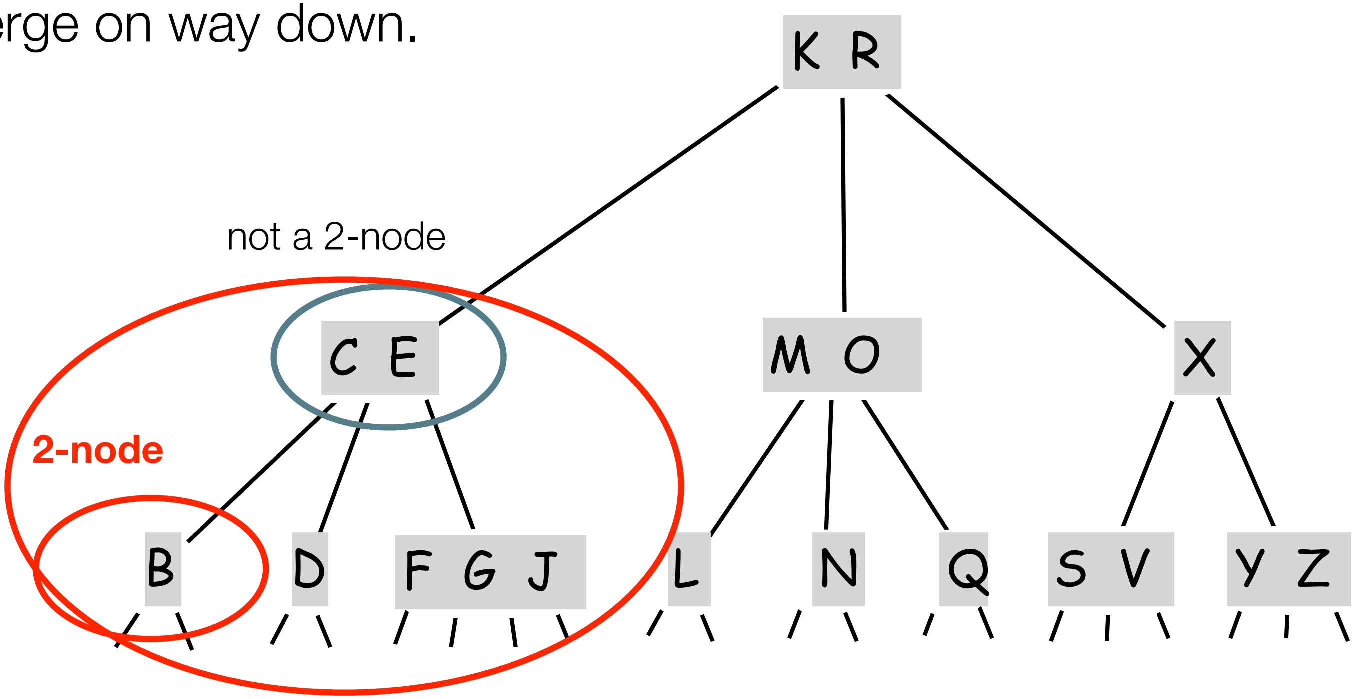


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node: split/merge on way down.

Ex. Delete minimum

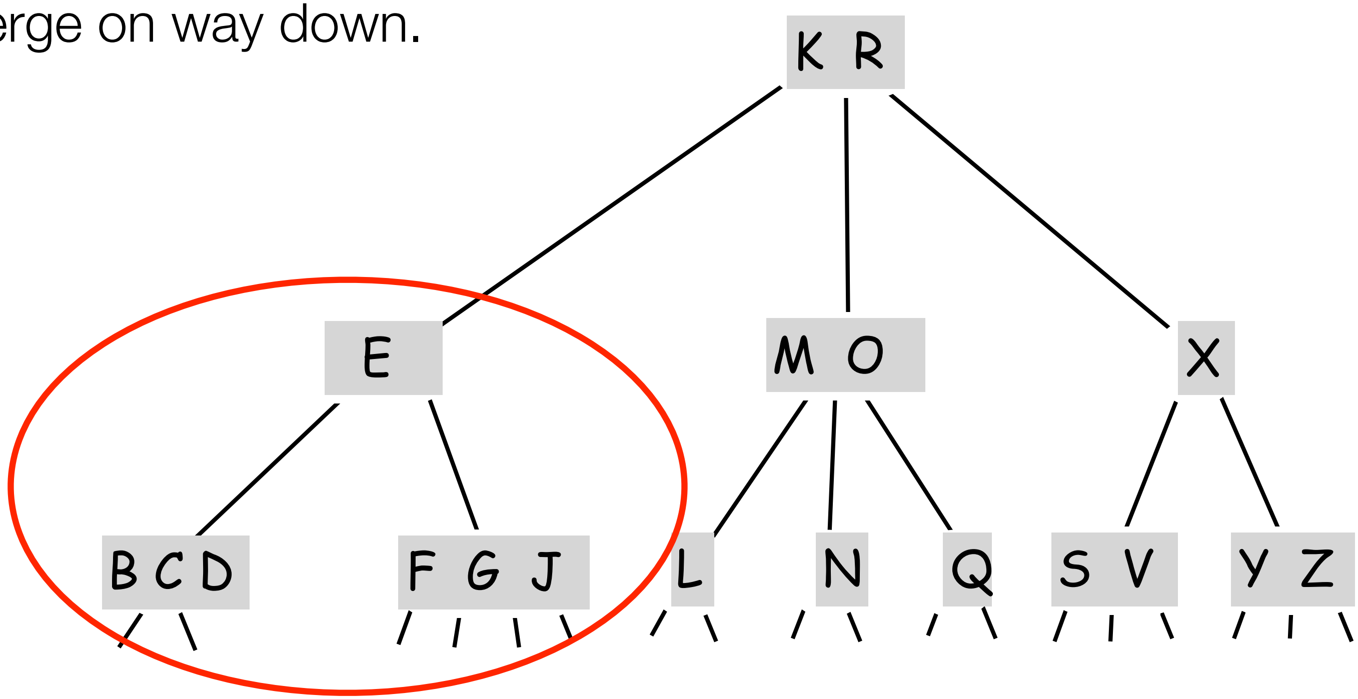


Deletions in 2-3-4 trees

Delete minimum:

- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node: split/merge on way down.

Ex. Delete minimum

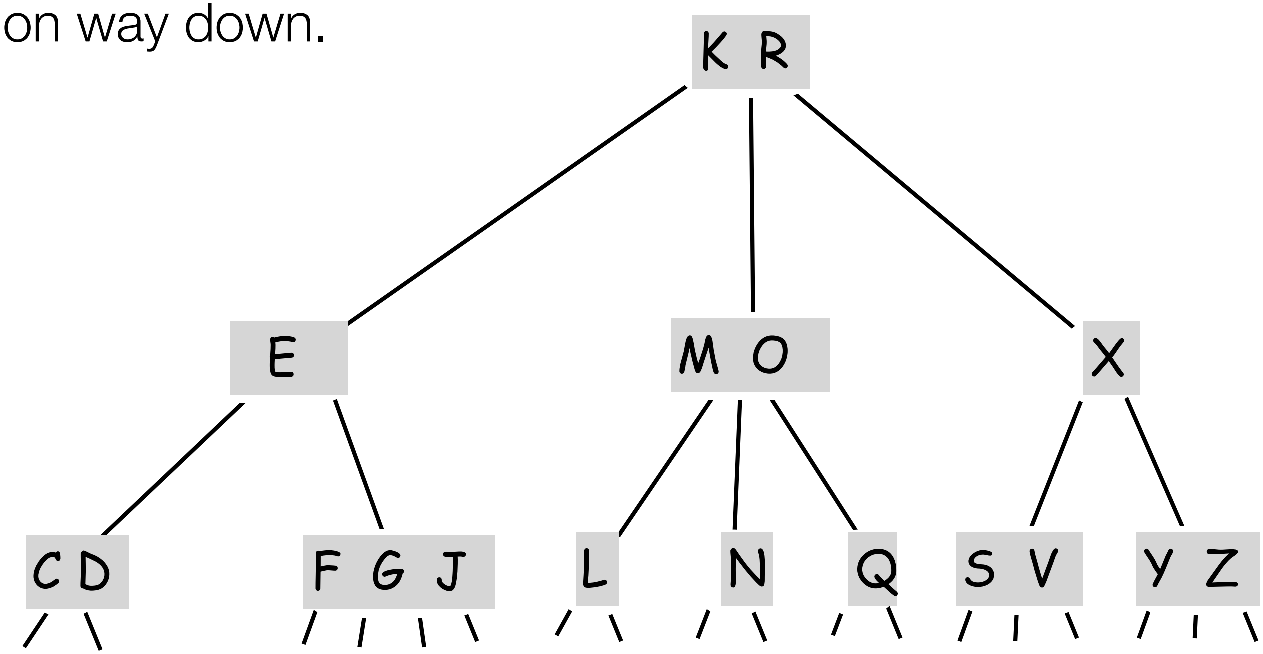


Deletions in 2-3-4 trees

Delete minimum:

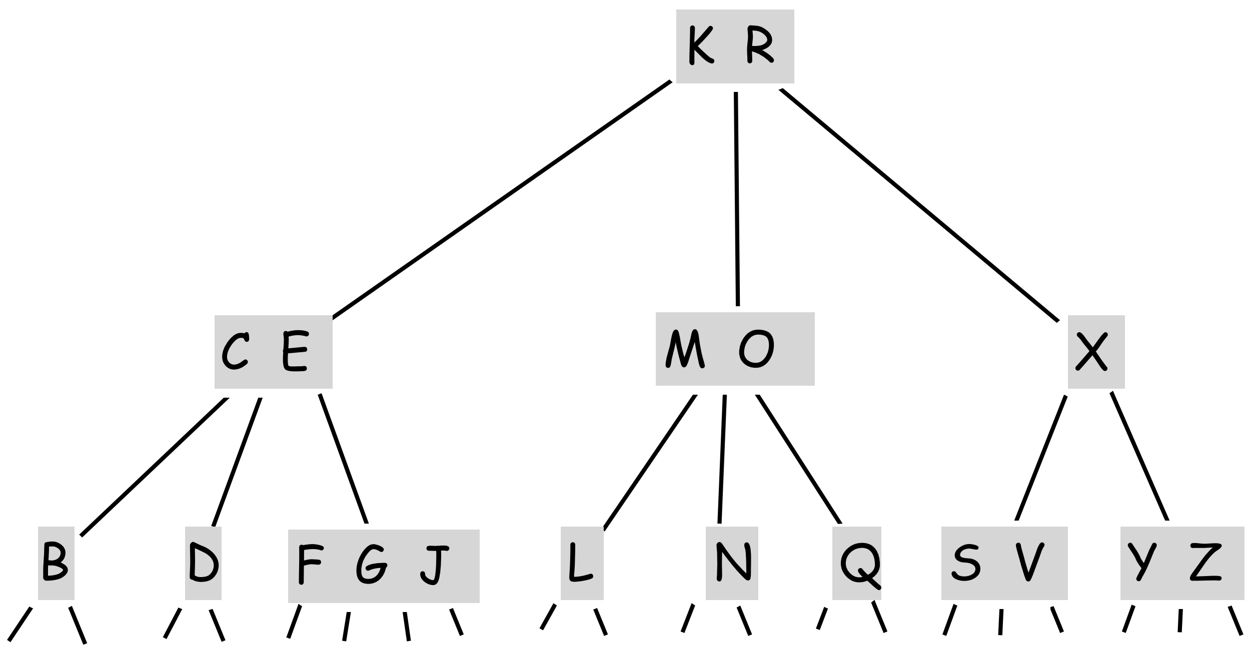
- minimum always in leftmost leaf
- If 3- or 4-node: delete key
- 2-node: split/merge on way down.

Ex. Delete minimum



Deletions in 2-3-4 trees

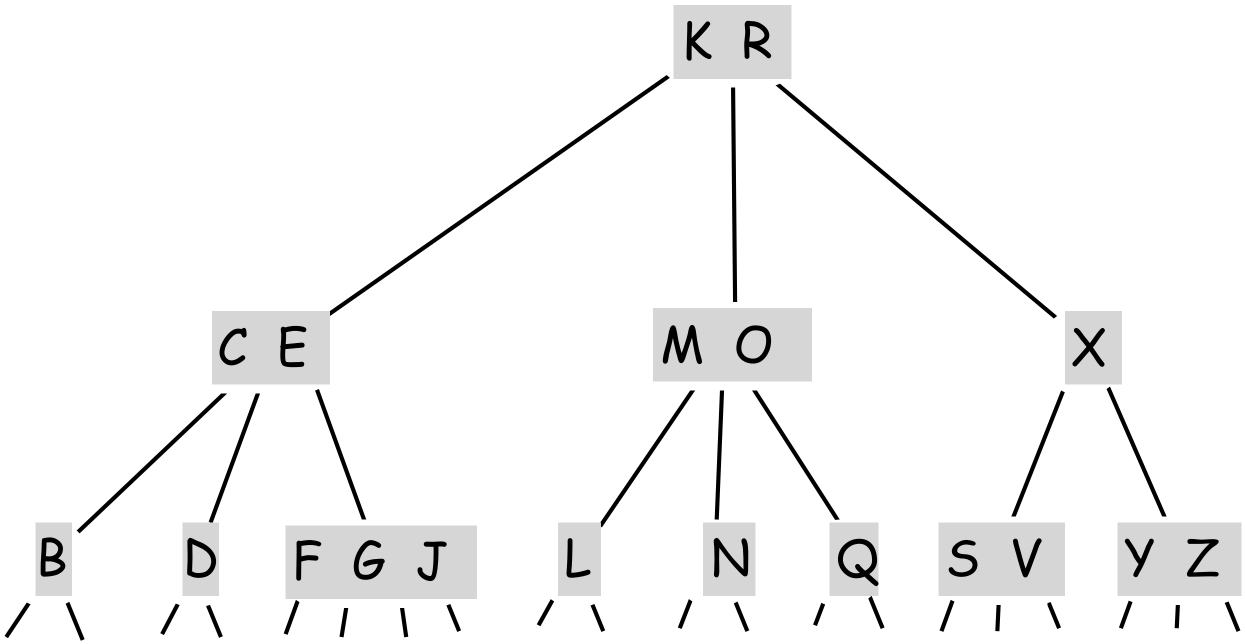
Delete:



Deletions in 2-3-4 trees

Delete:

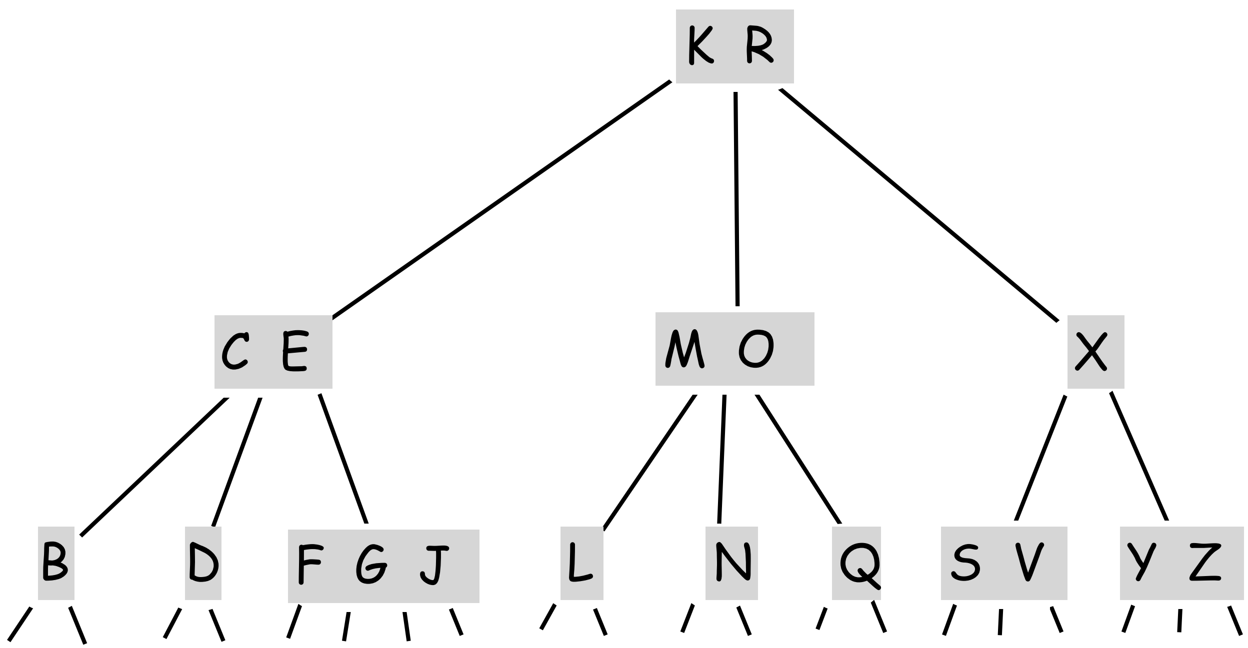
- During search maintain invariant that current node is not a 2-node



Deletions in 2-3-4 trees

Delete:

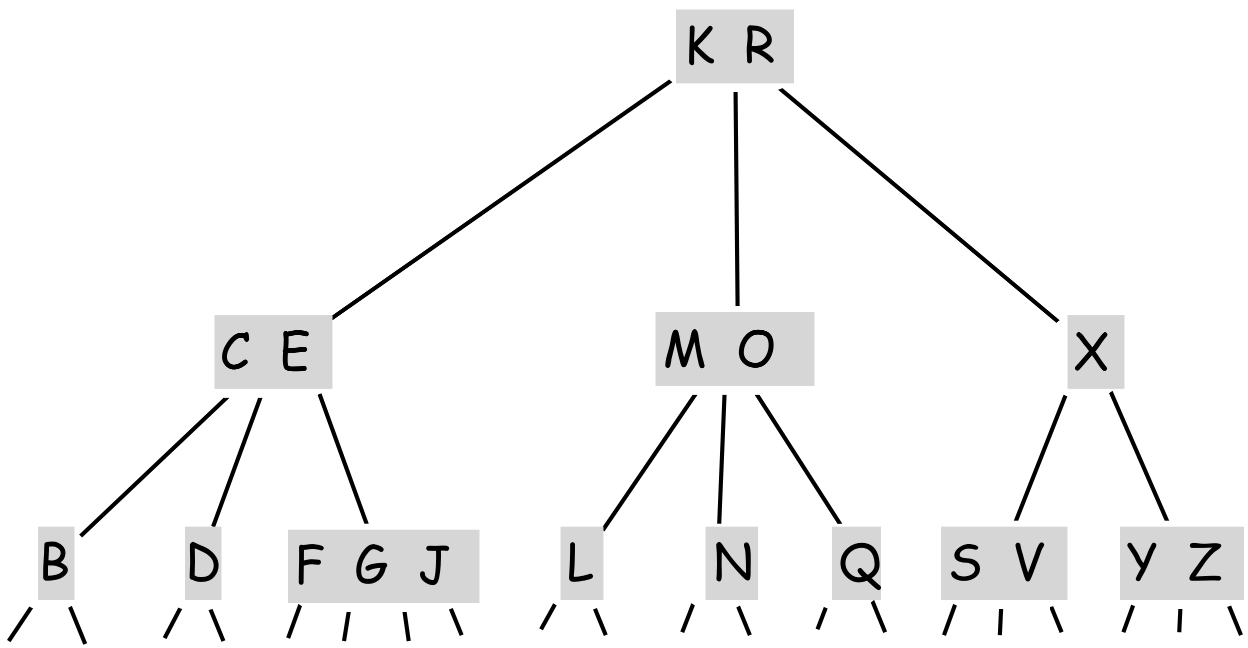
- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key



Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

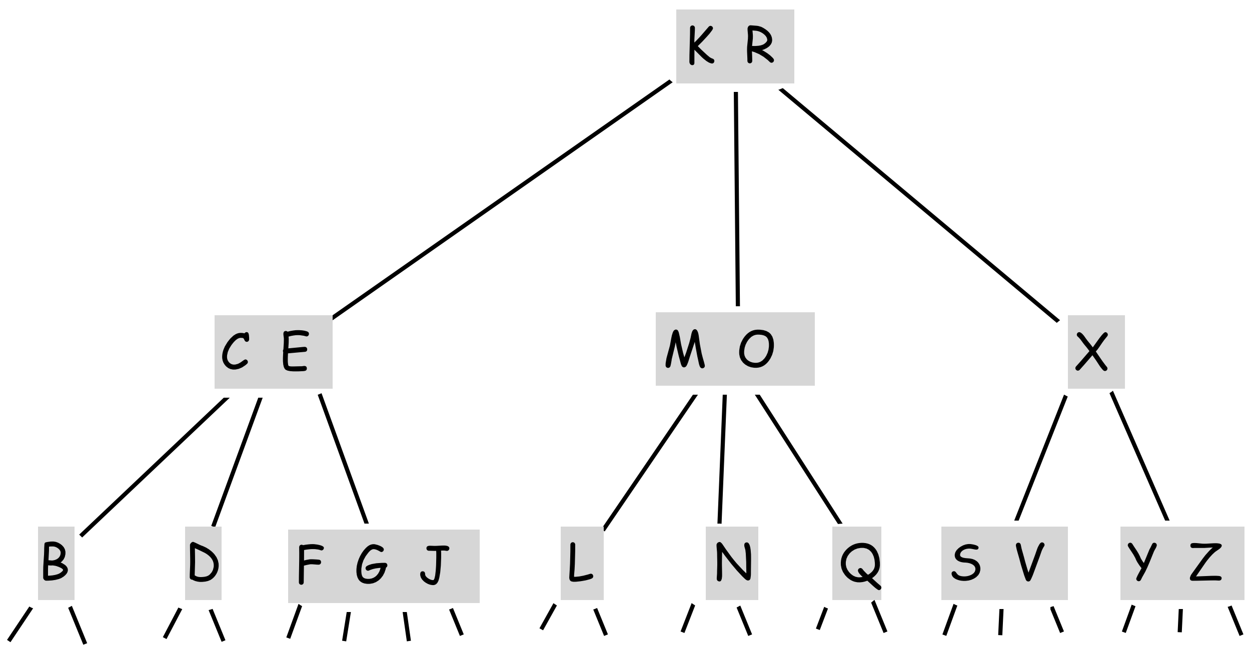


Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K



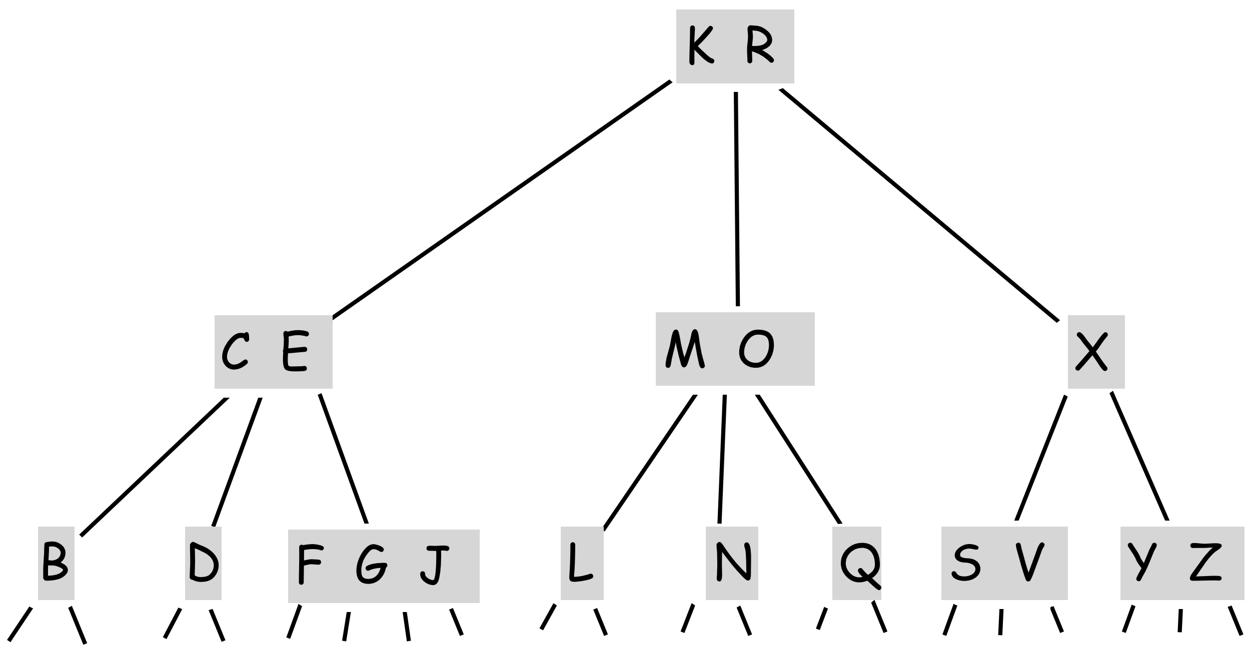
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor



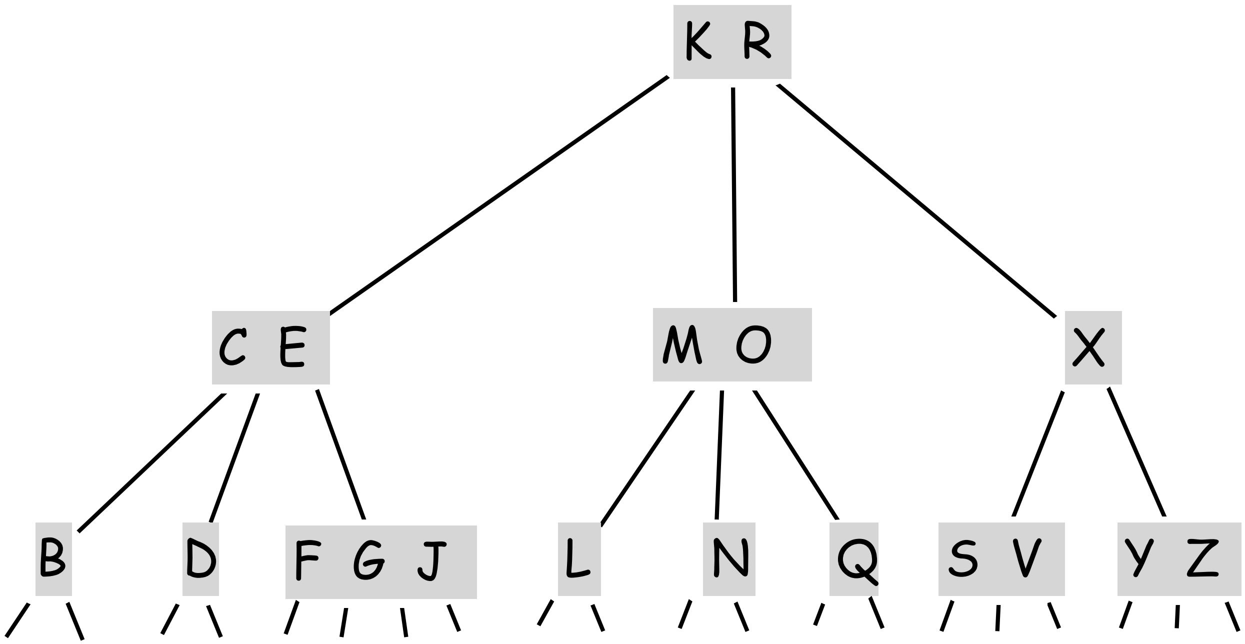
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor



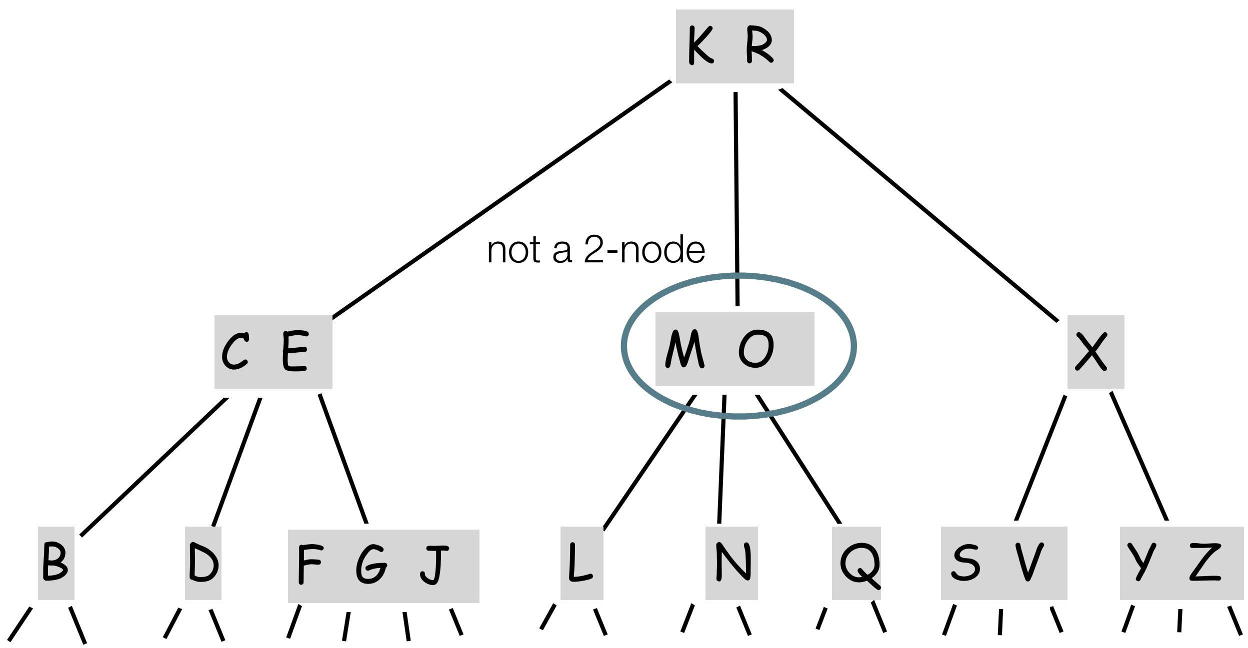
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor



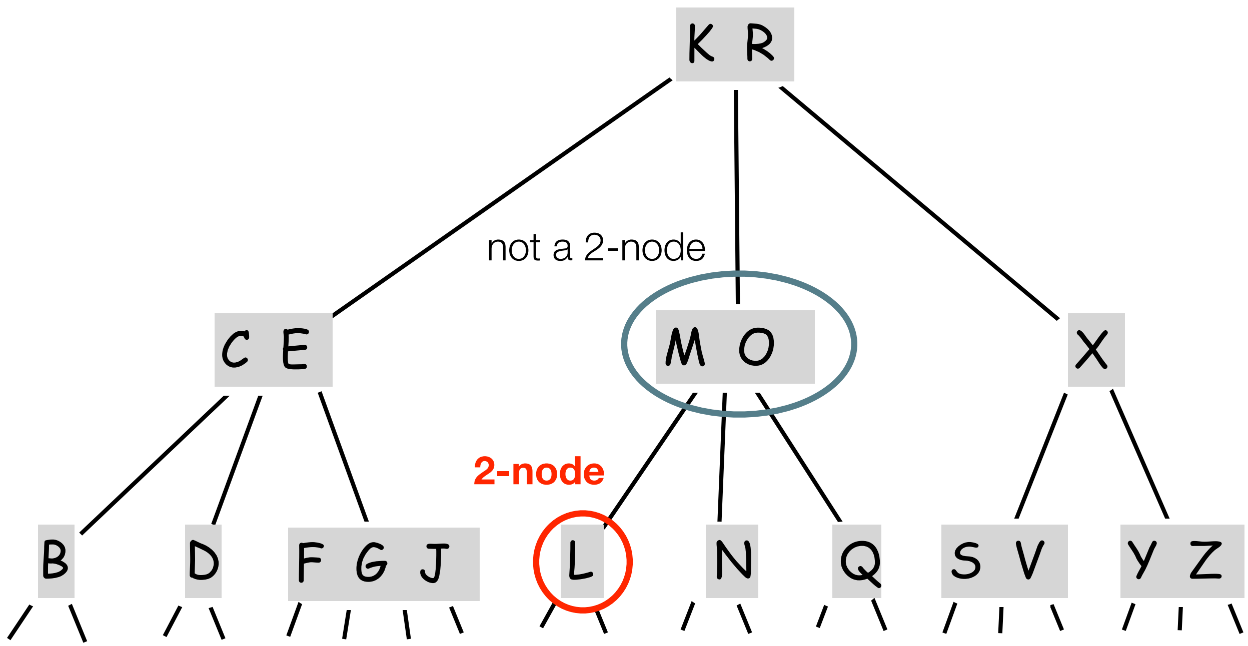
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor



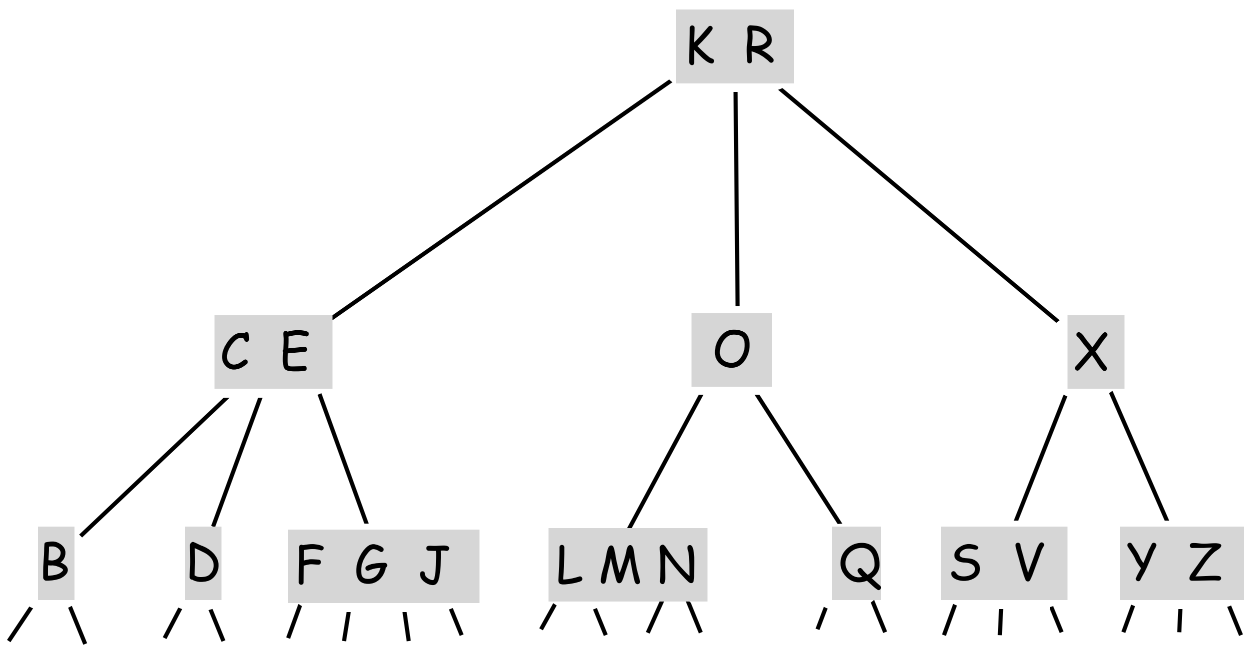
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor



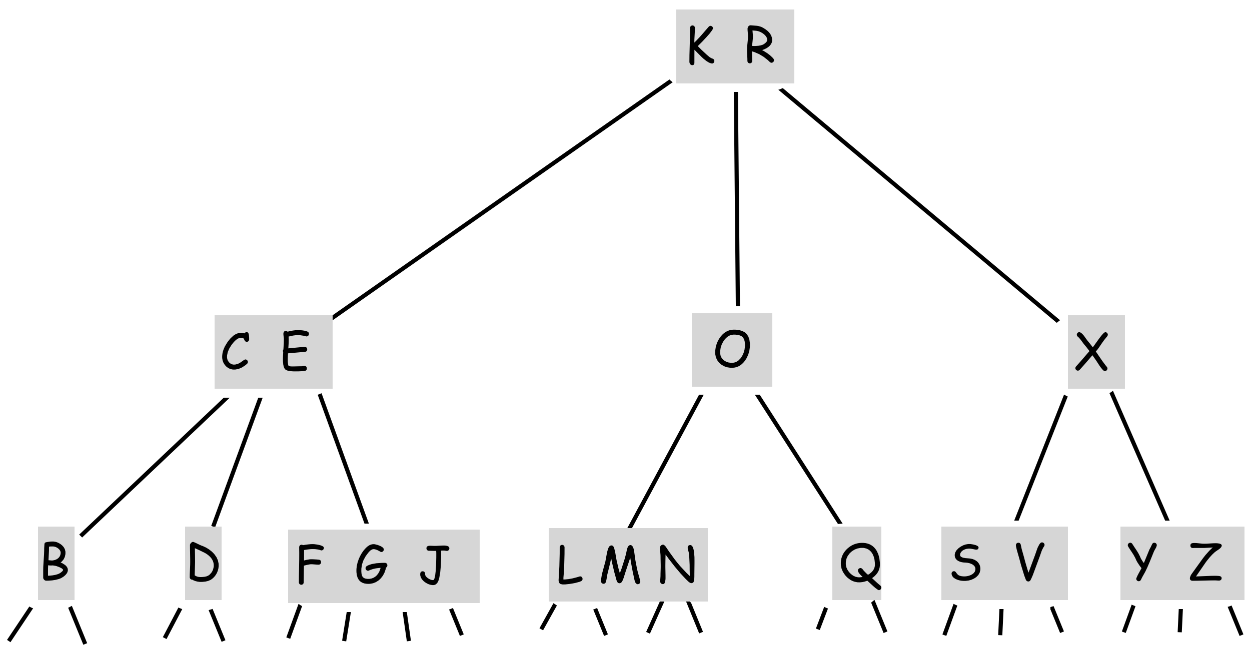
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor
- Delete L from leaf



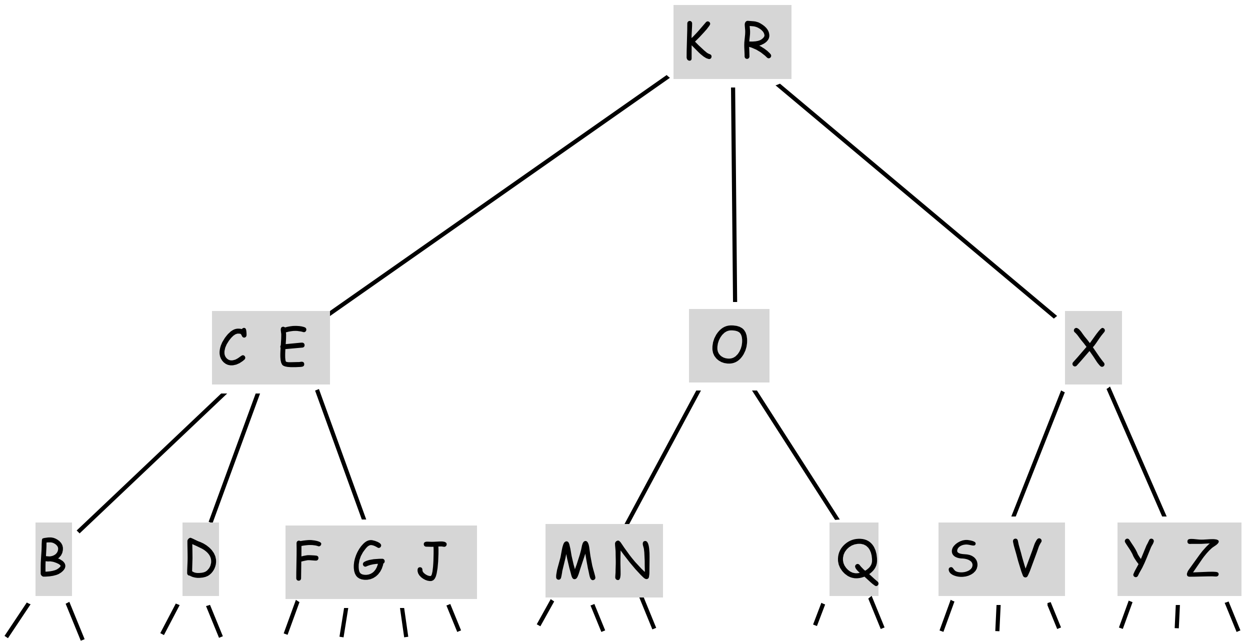
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor
- Delete L from leaf



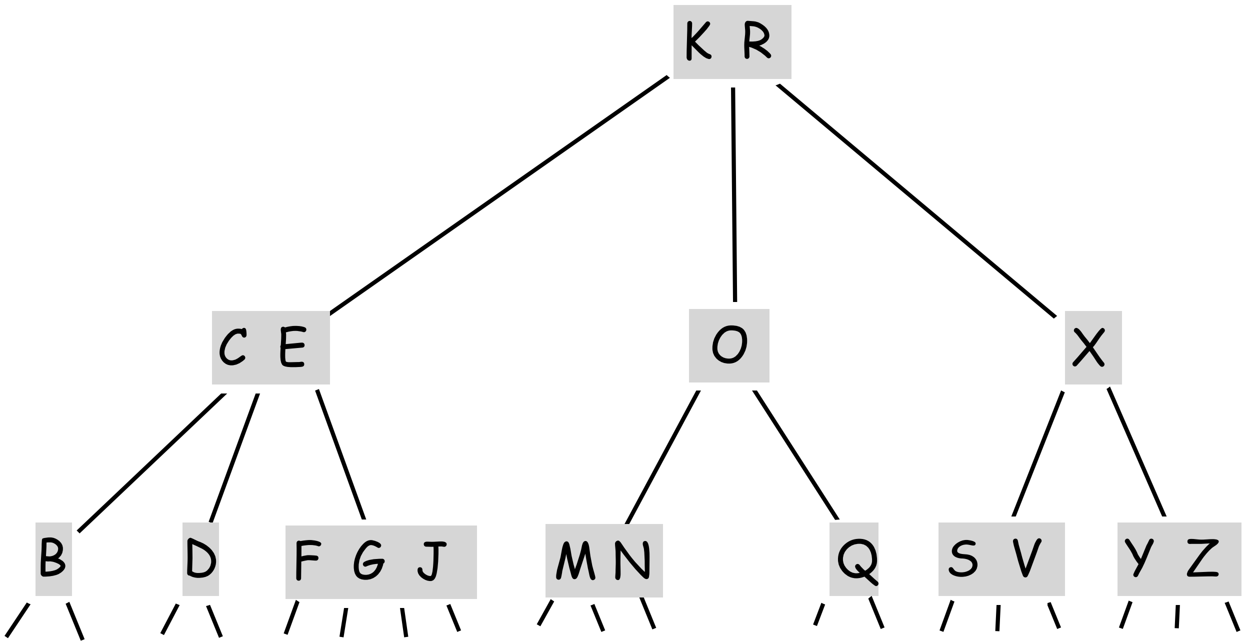
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

Ex. Delete K

- Find successor
- Delete L from leaf
- Replace K with L



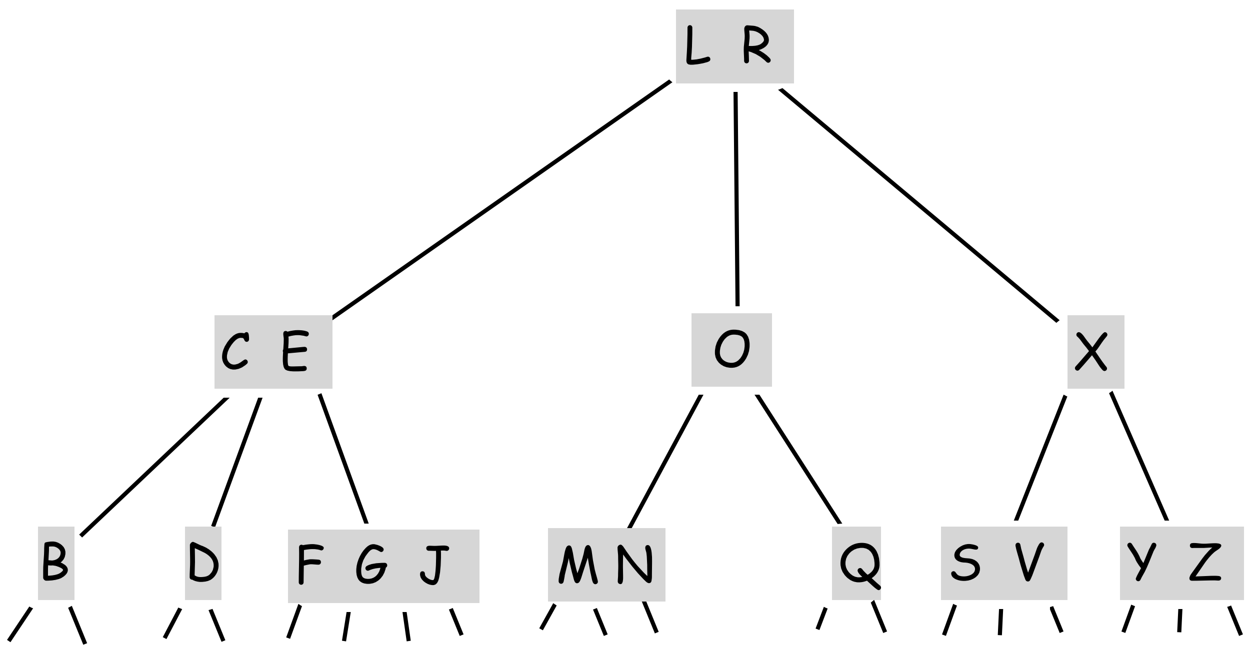
Deletions in 2-3-4 trees

Delete:

- During search maintain invariant that current node is not a 2-node
- If key is in a leaf: delete key
- Key not in leaf: replace with successor (always leaf in subtree) and delete successor from leaf.

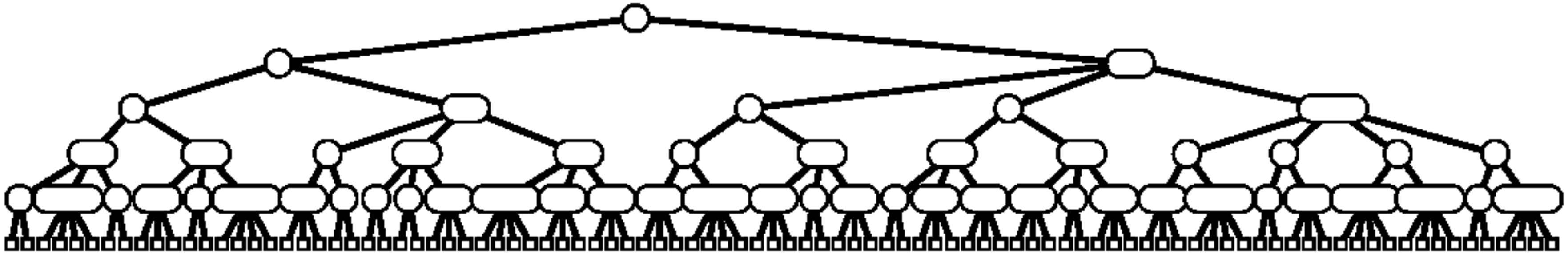
Ex. Delete K

- Find successor
- Delete L from leaf
- Replace K with L



2-3-4 Tree: Balance

Property. All paths from root to leaf have same length.



Tree height.

Worst case: lg N [all 2-nodes]

Best case: $\log_4 N = 1/2 \lg N$ [all 4-nodes]

Between 10 and 20 for a million nodes.

Between 15 and 30 for a billion nodes.

Dynamic set implementations

Worst case running times

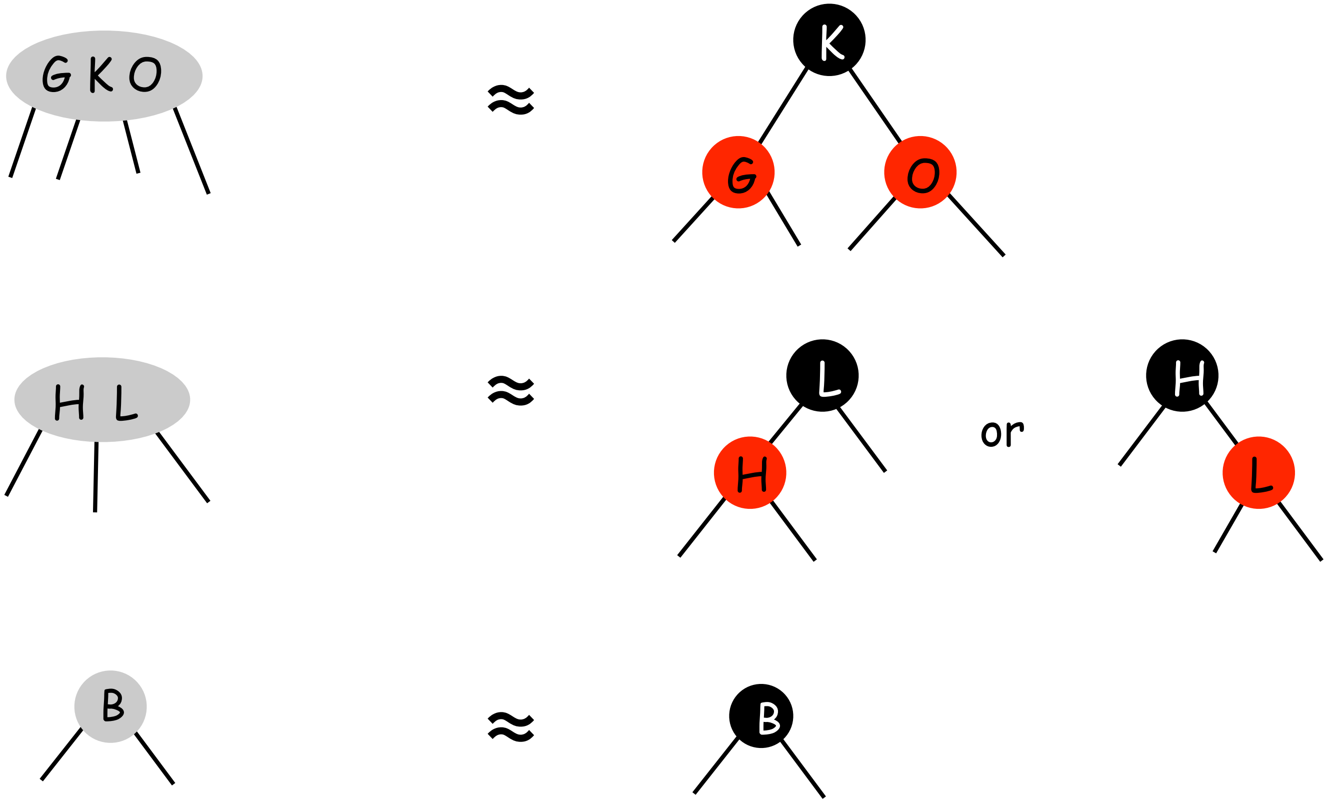
Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)
2-3-4 tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)

Red-black trees

Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

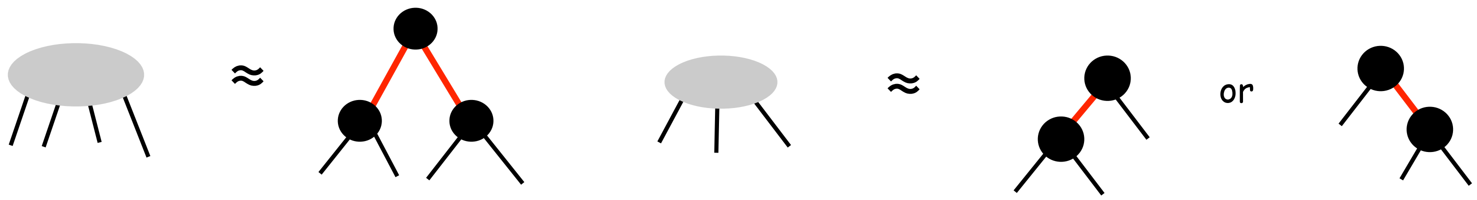
- Use colors on nodes to represent 3- and 4-nodes.



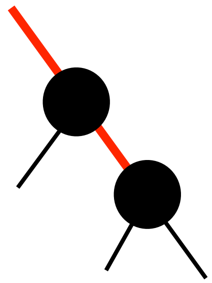
Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

- Use colors on nodes to represent 3- and 4-nodes.



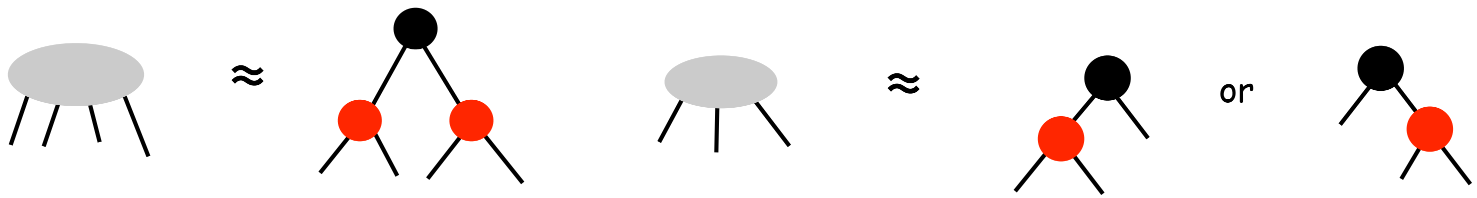
- Disallowed: 2 red nodes in-a-row.



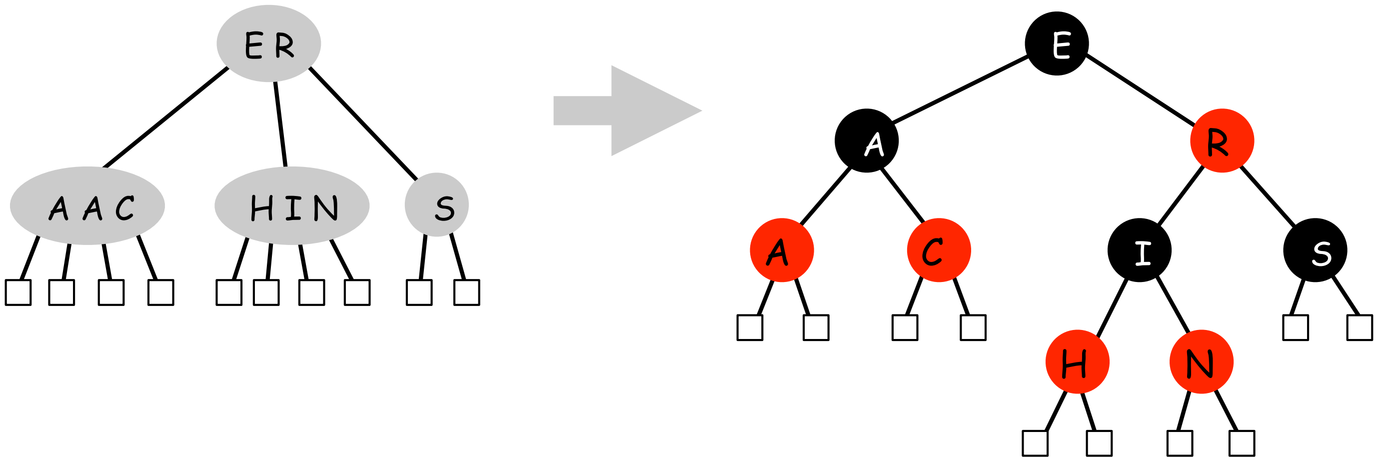
Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

- Use colors on nodes to represent 3- and 4-nodes.



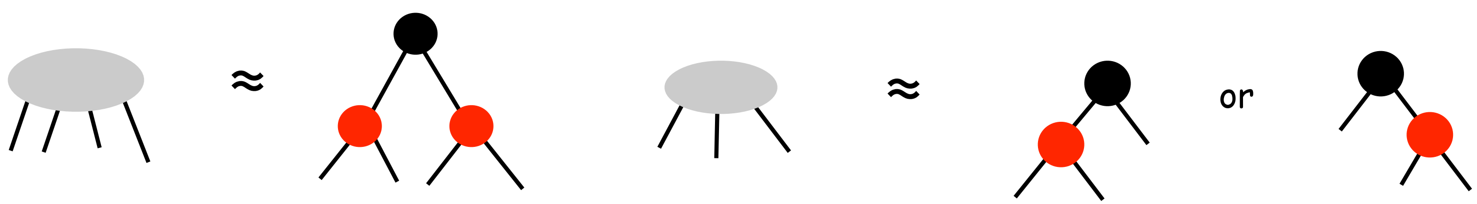
- Connection between 2-3-4 trees and red-black trees:



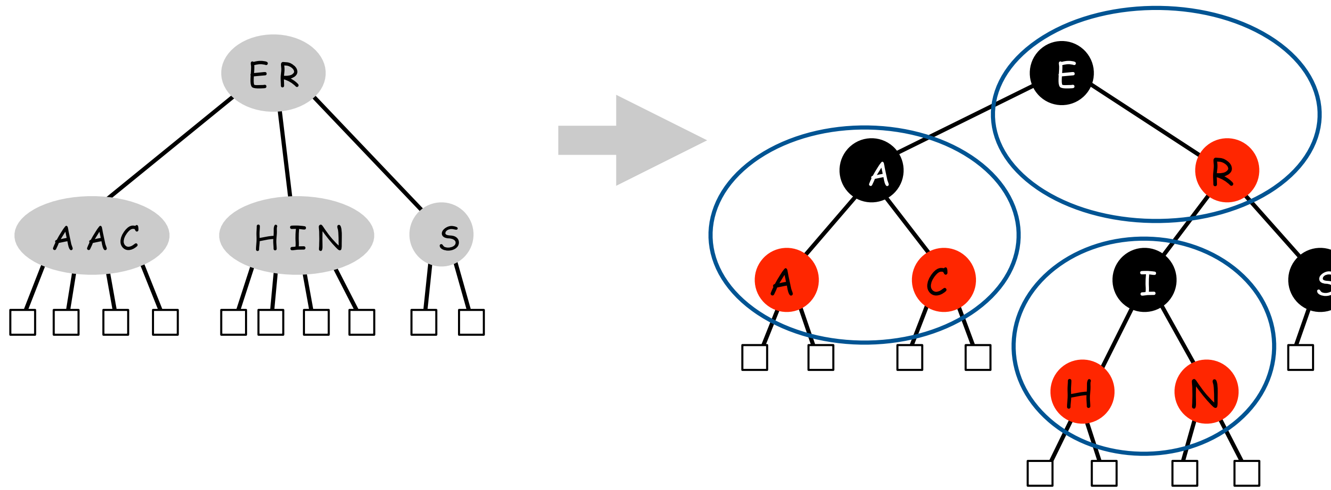
Red-black tree (Guibas-Sedgewick, 1979)

Represent 2-3-4 tree as a binary search tree

- Use colors on nodes to represent 3- and 4-nodes.



- Connection between 2-3-4 trees and red-black trees:



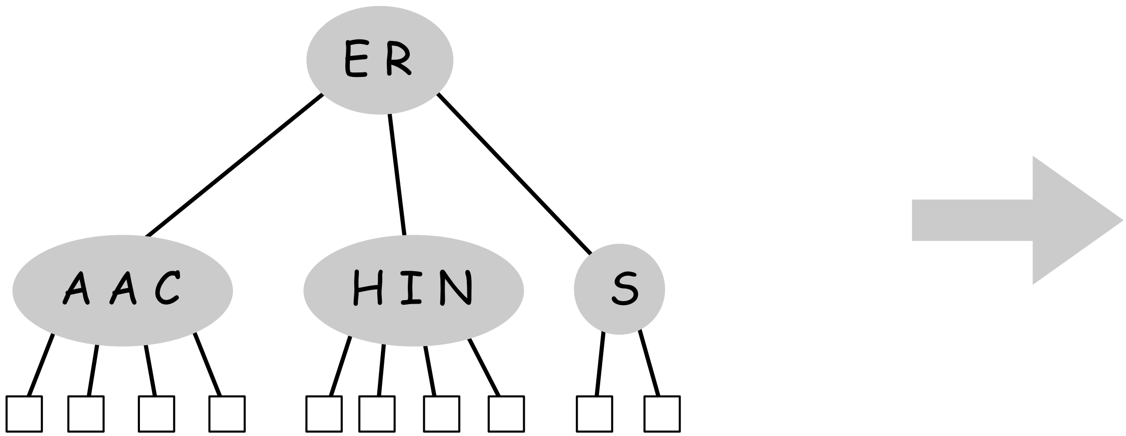
Red-black tree

Properties of red-black trees:

- The root is always black
- All root-to-leaf paths have the same number of black nodes.
- Red nodes do not have red children
- CLRS: All leaves (NIL) are black

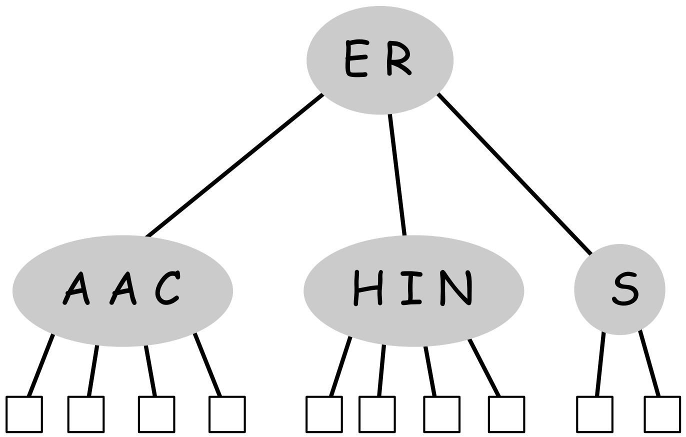
Red-black tree

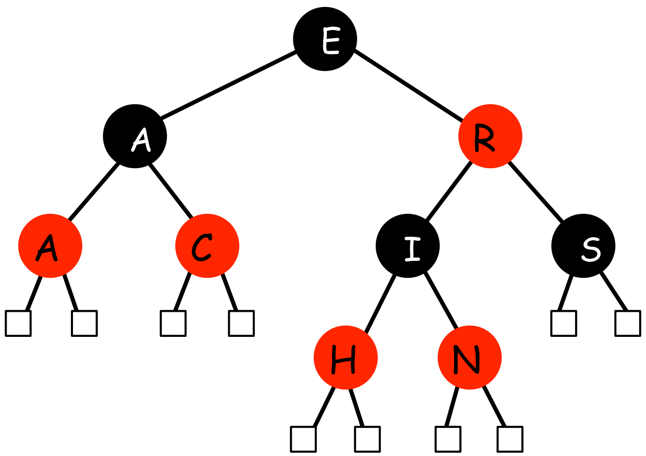
Connection between 2-3-4 trees and red-black trees:



Red-black tree

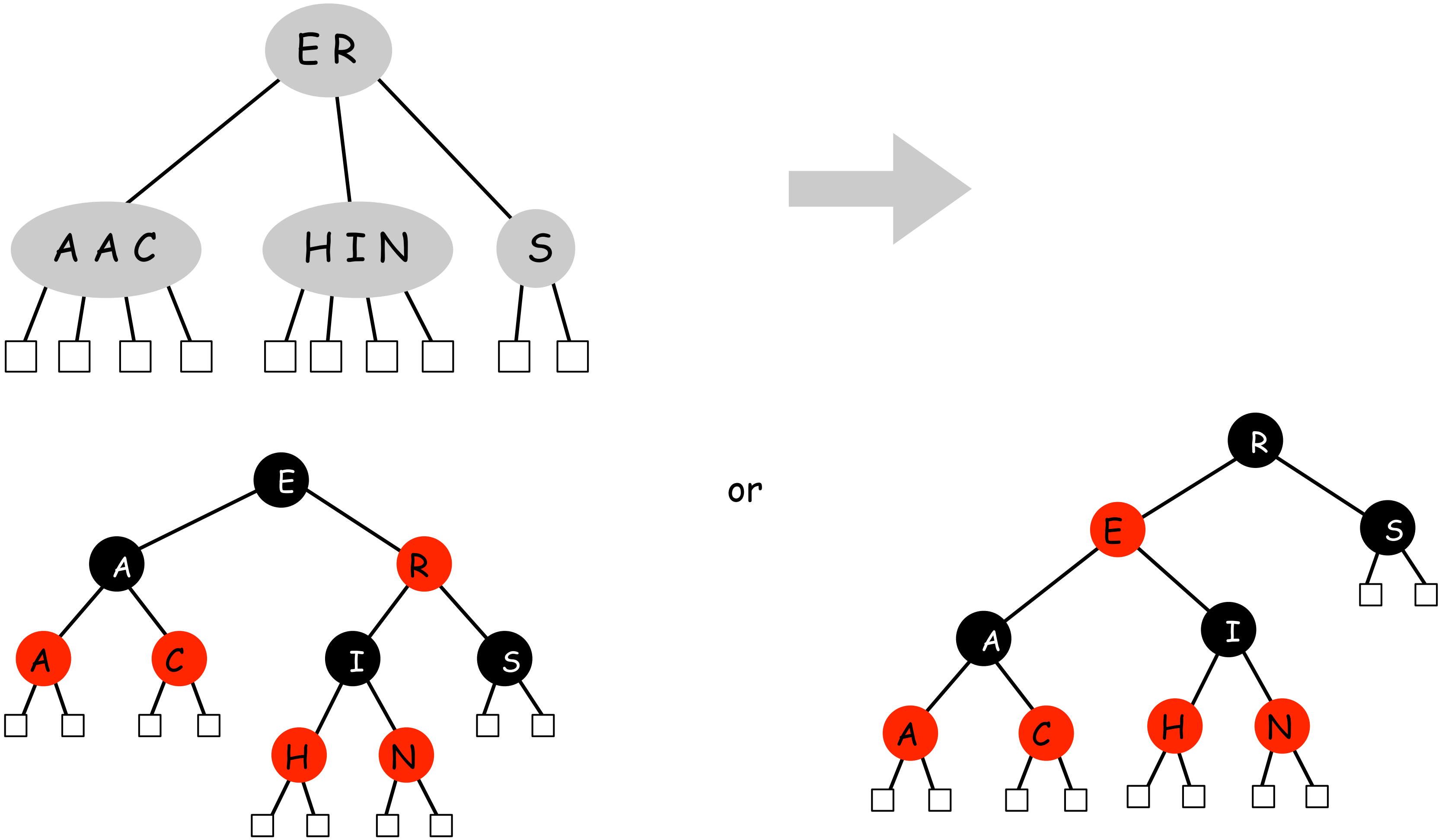
Connection between 2-3-4 trees and red-black trees:





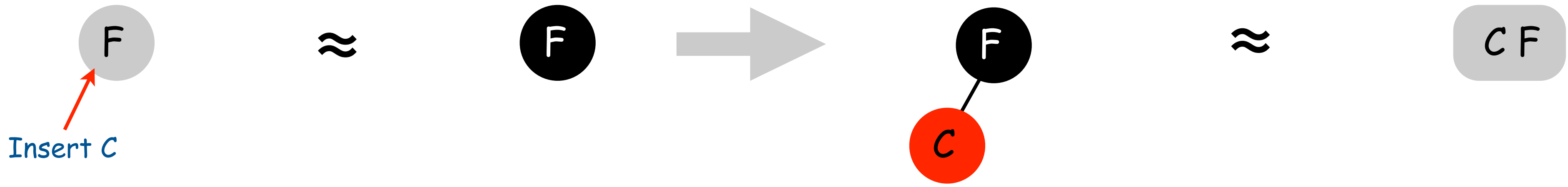
Red-black tree

Connection between 2-3-4 trees and red-black trees:

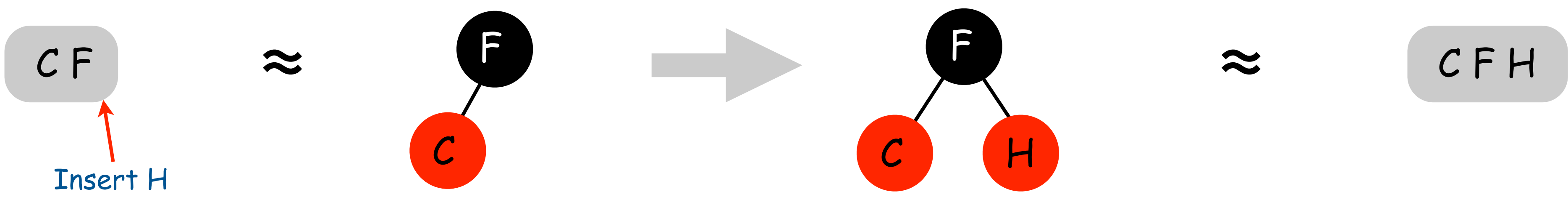


Insertion in red-black trees

Insertion in 2-node:

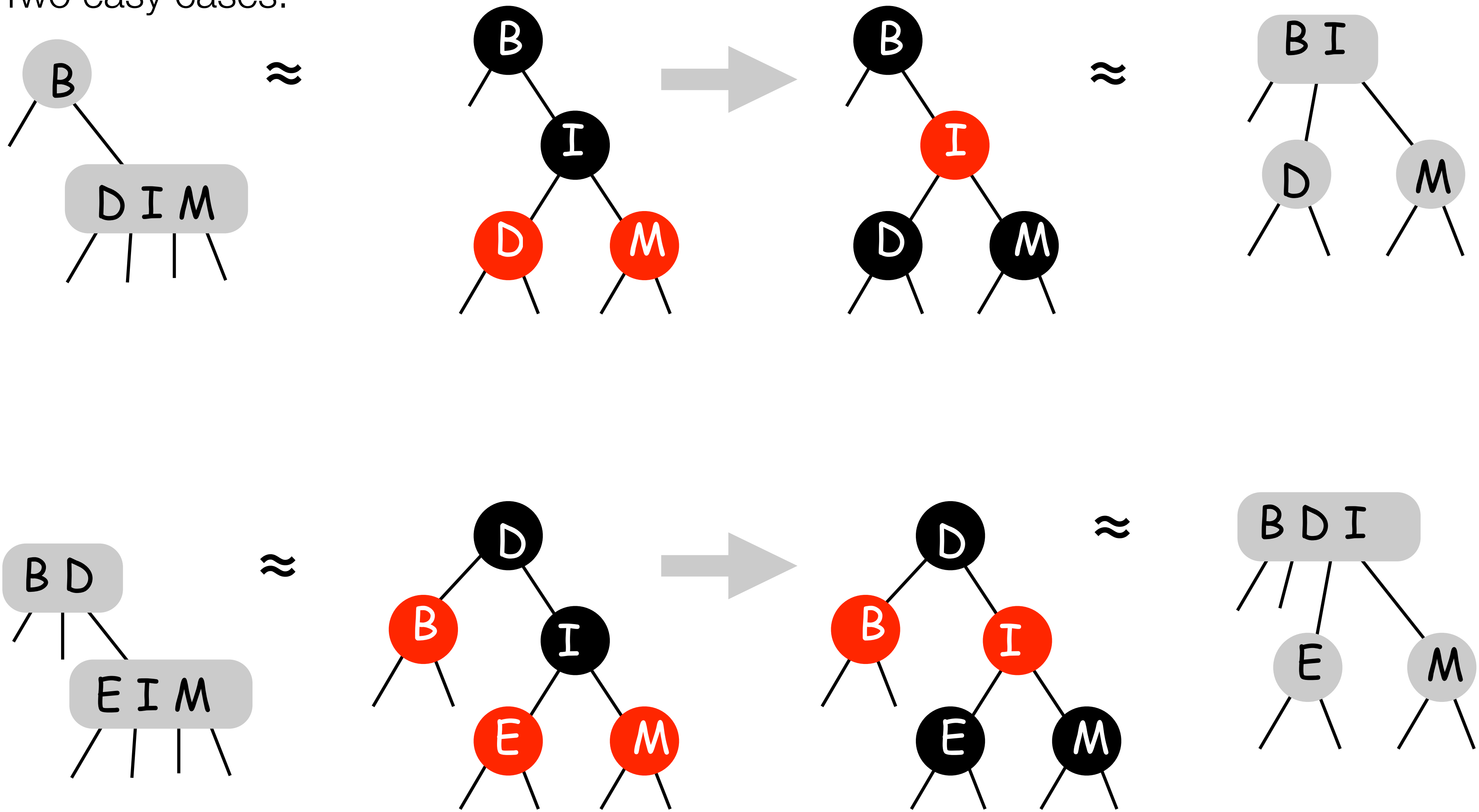


Insertion in 3-node:



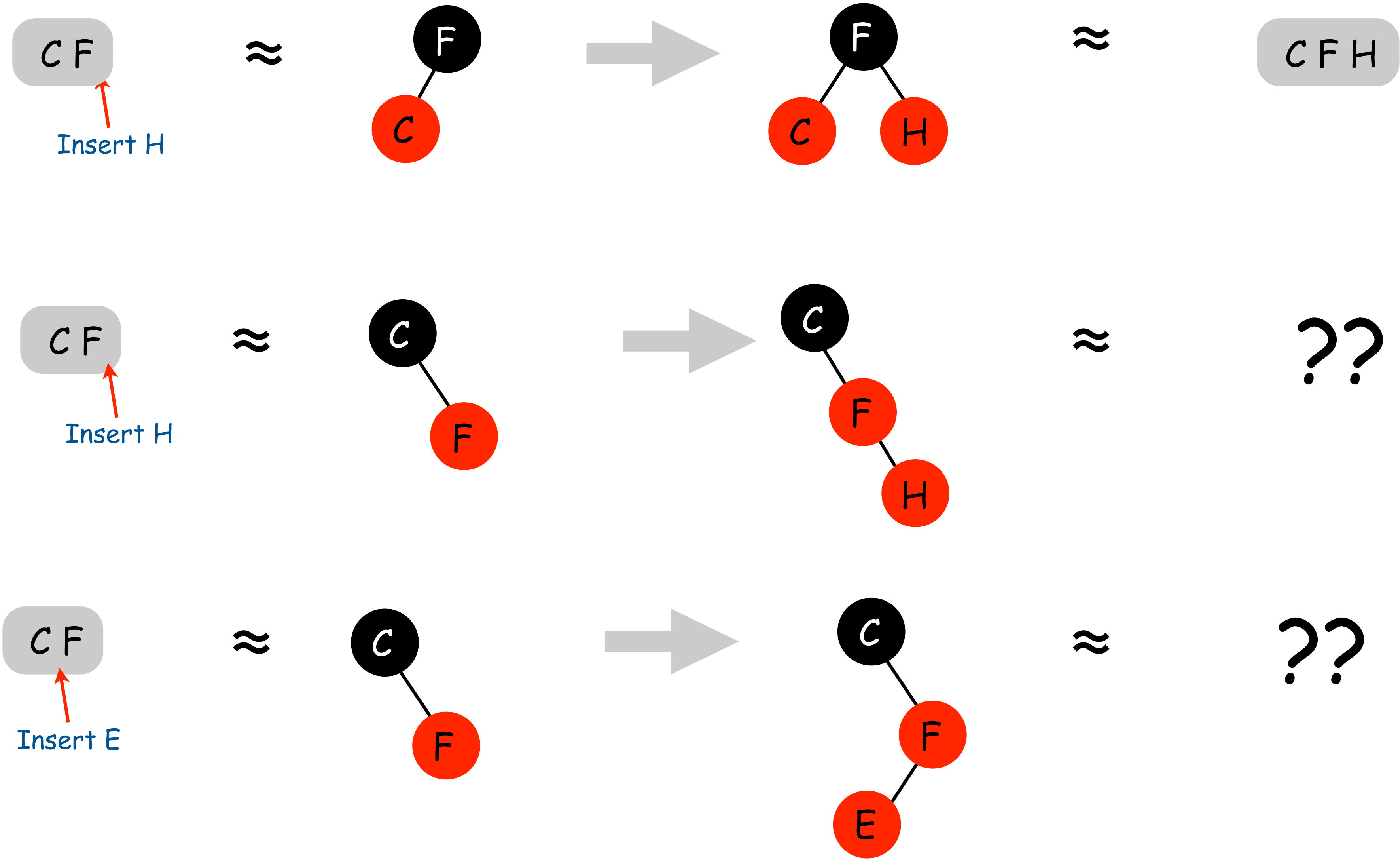
Red-black tree: Splitting 4-nodes

Two easy cases:



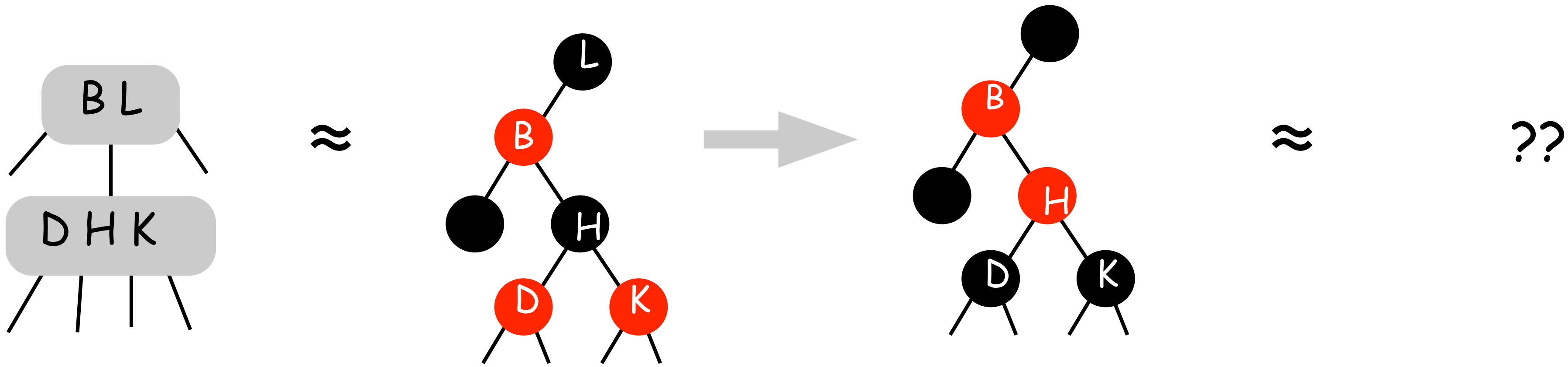
Insertion in red-black trees

Insertion in 3-node (continued):



Red-black trees: Splitting of 4-nodes

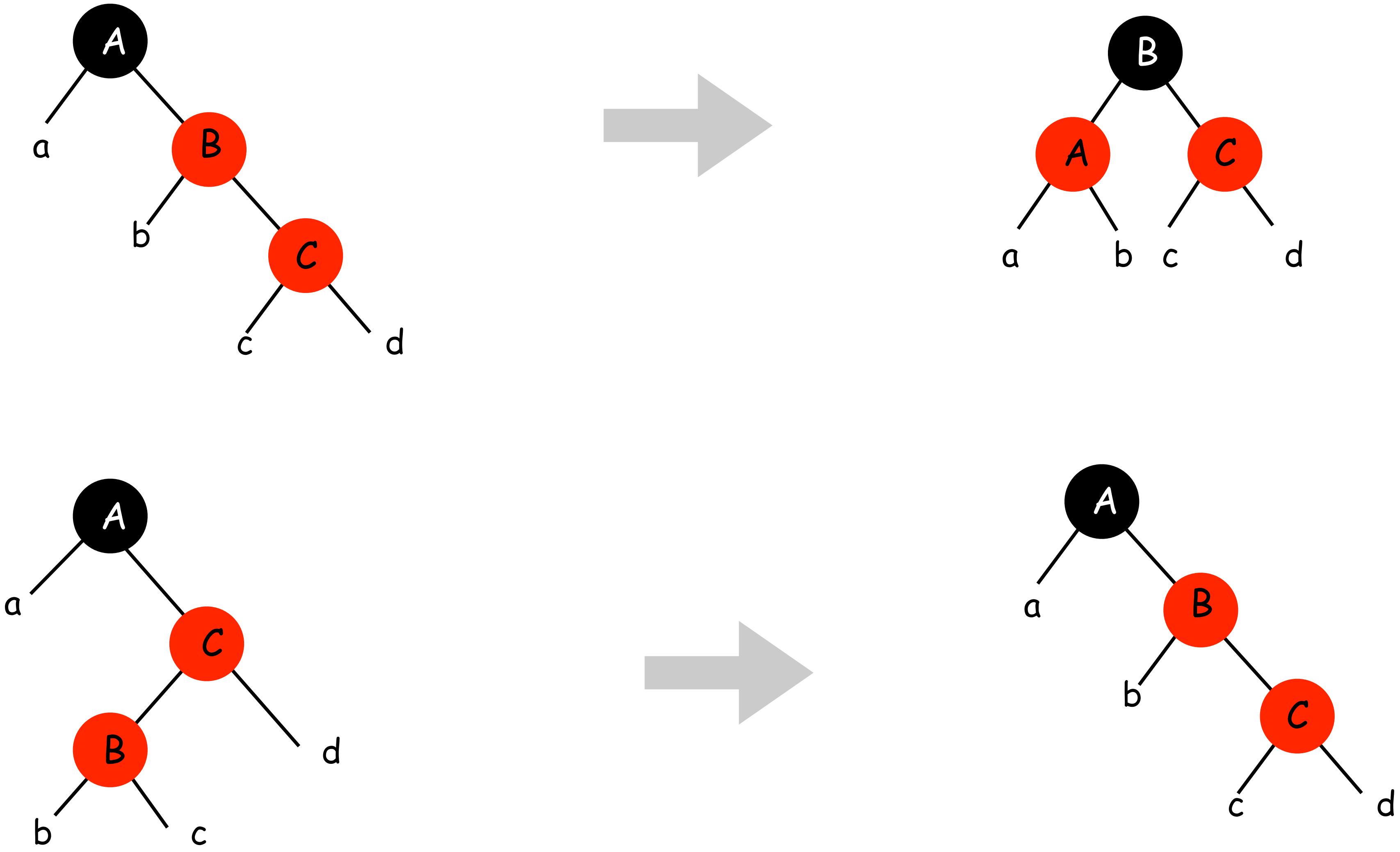
Example of hard case:



Solution: Rotations!

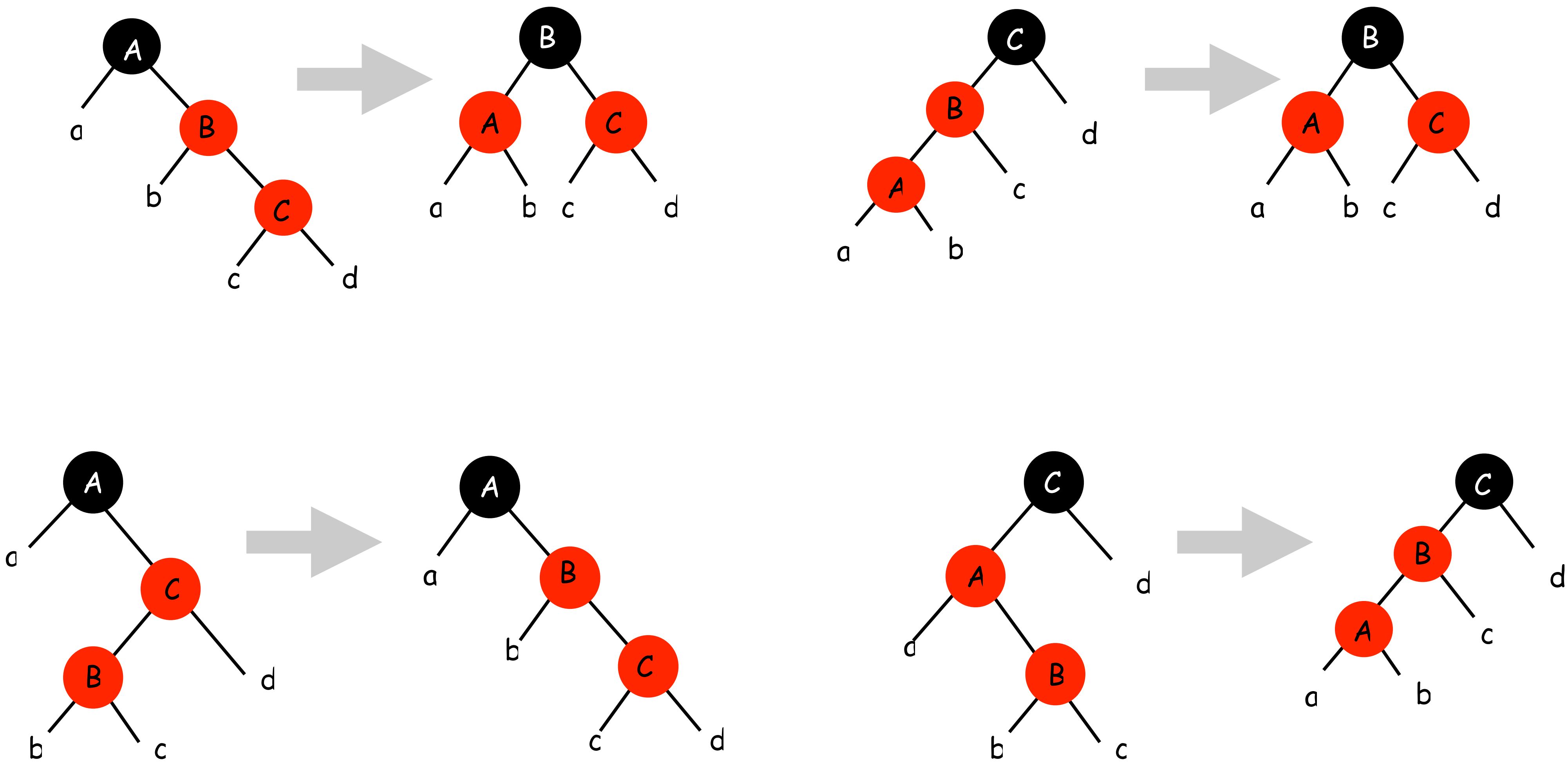
Rotations in red-black trees

Two types of rotations



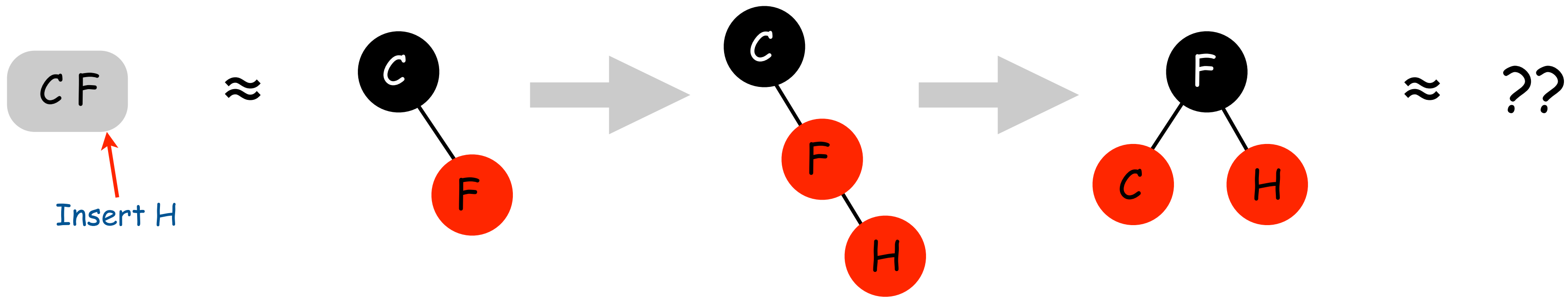
Rotations in red-black trees

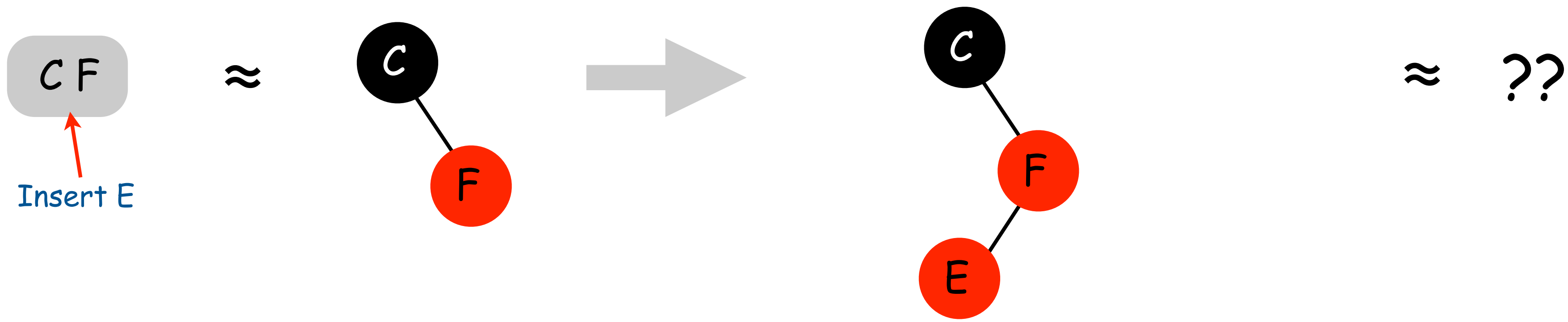
Two types of rotations:



Insertion in red-black trees

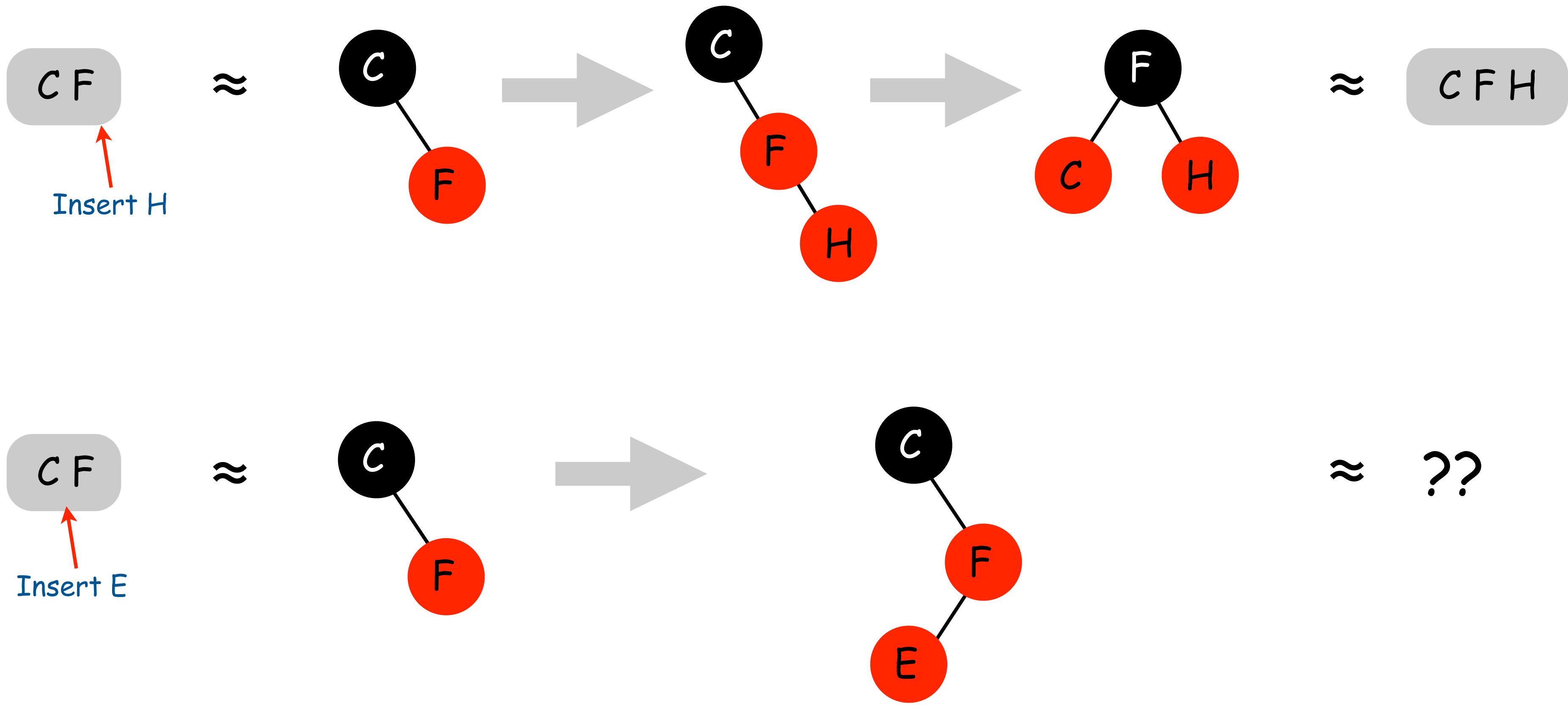
Insertion in 3-node:





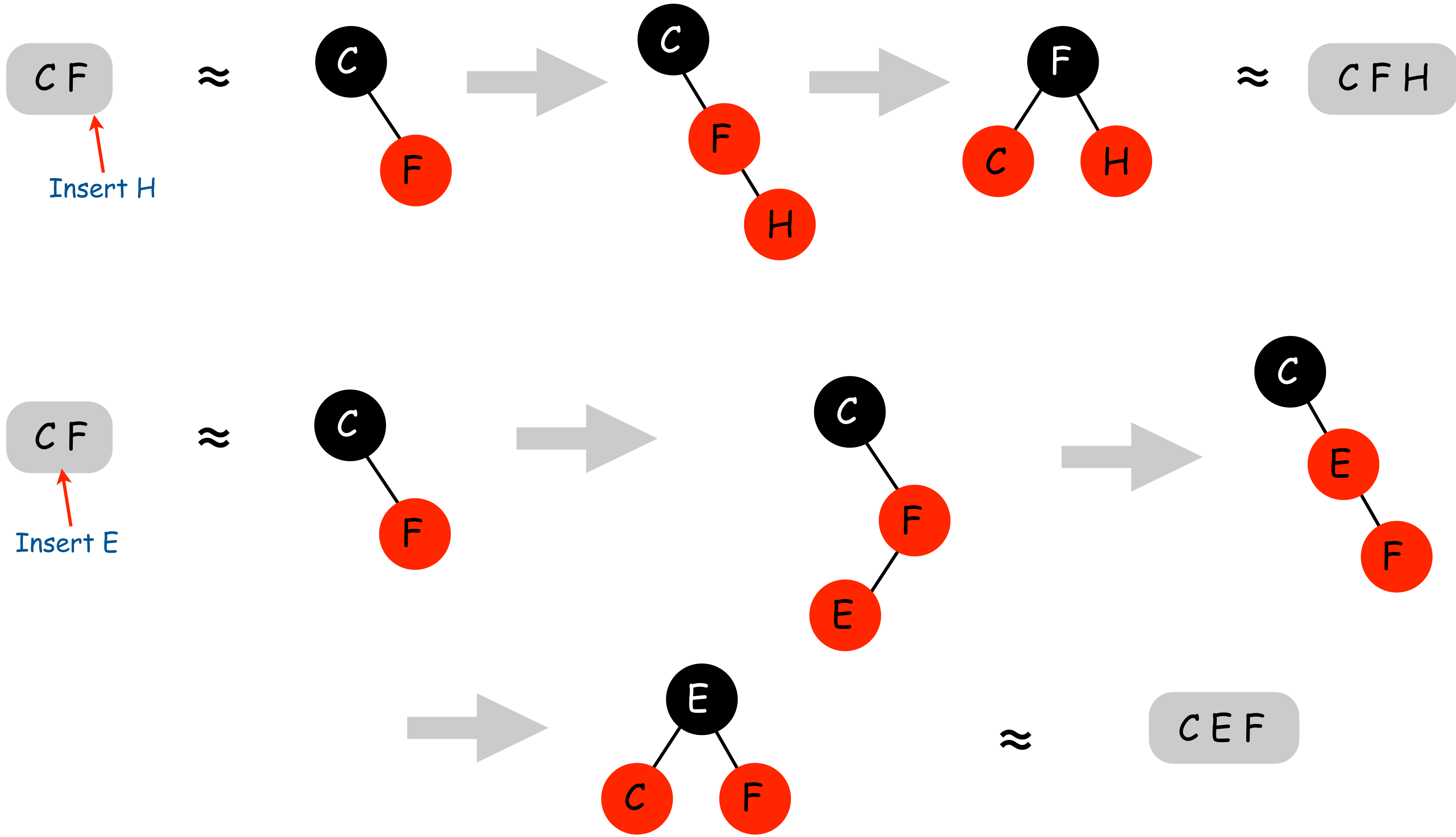
Insertion in red-black trees

Insertion in 3-node:



Insertion in red-black trees

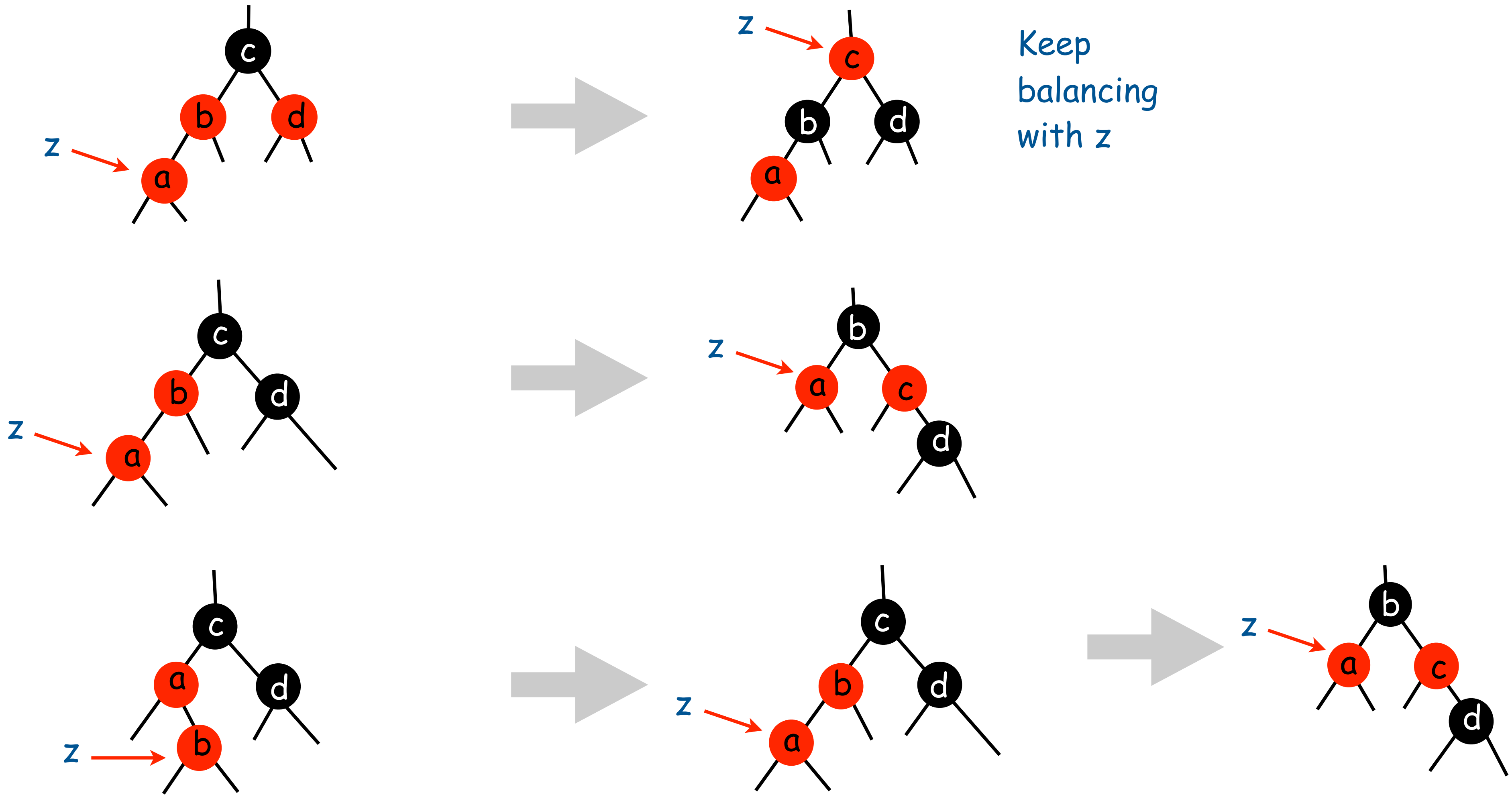
Insertion in 3-node:



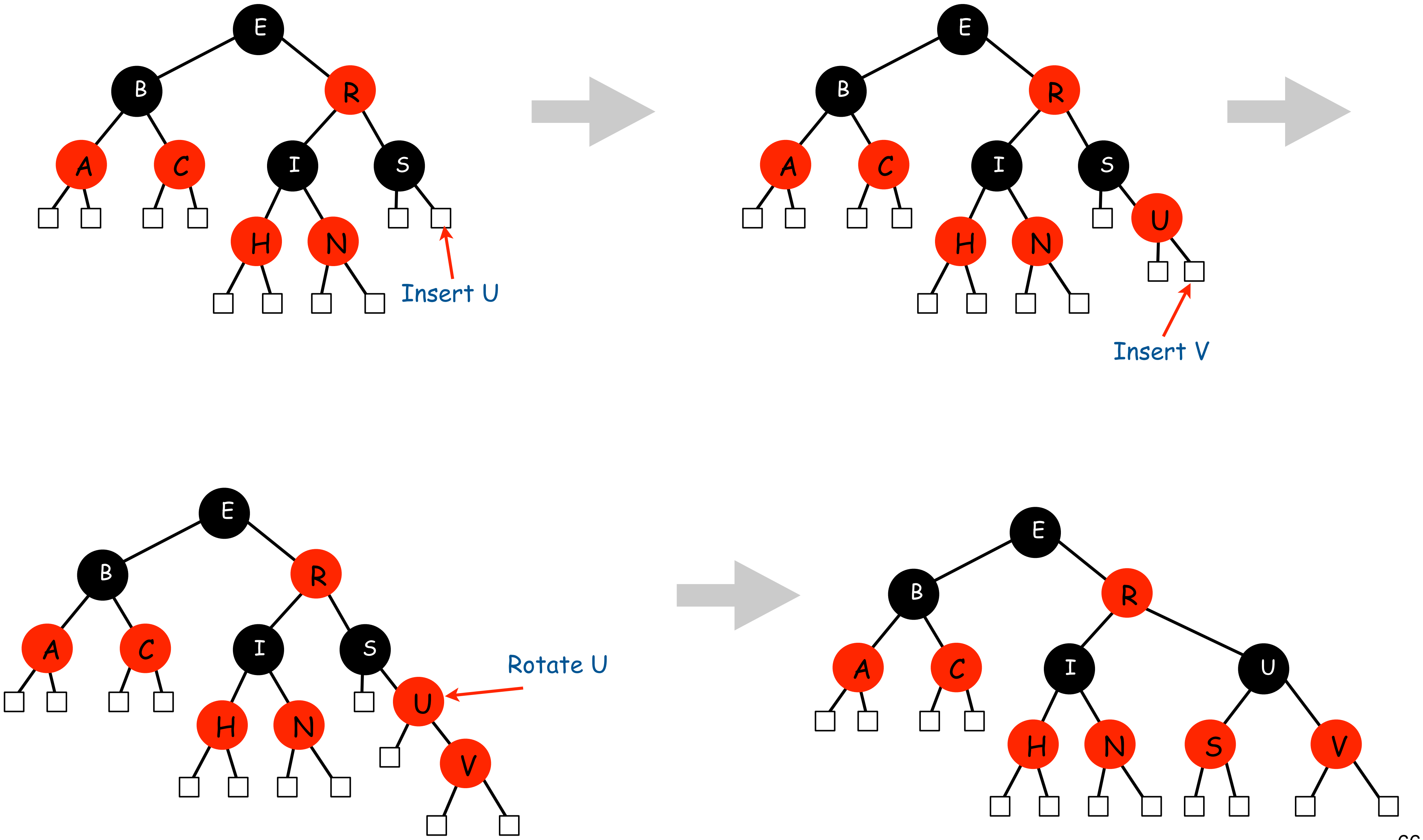
Insertion in red-black tree

Insert x:

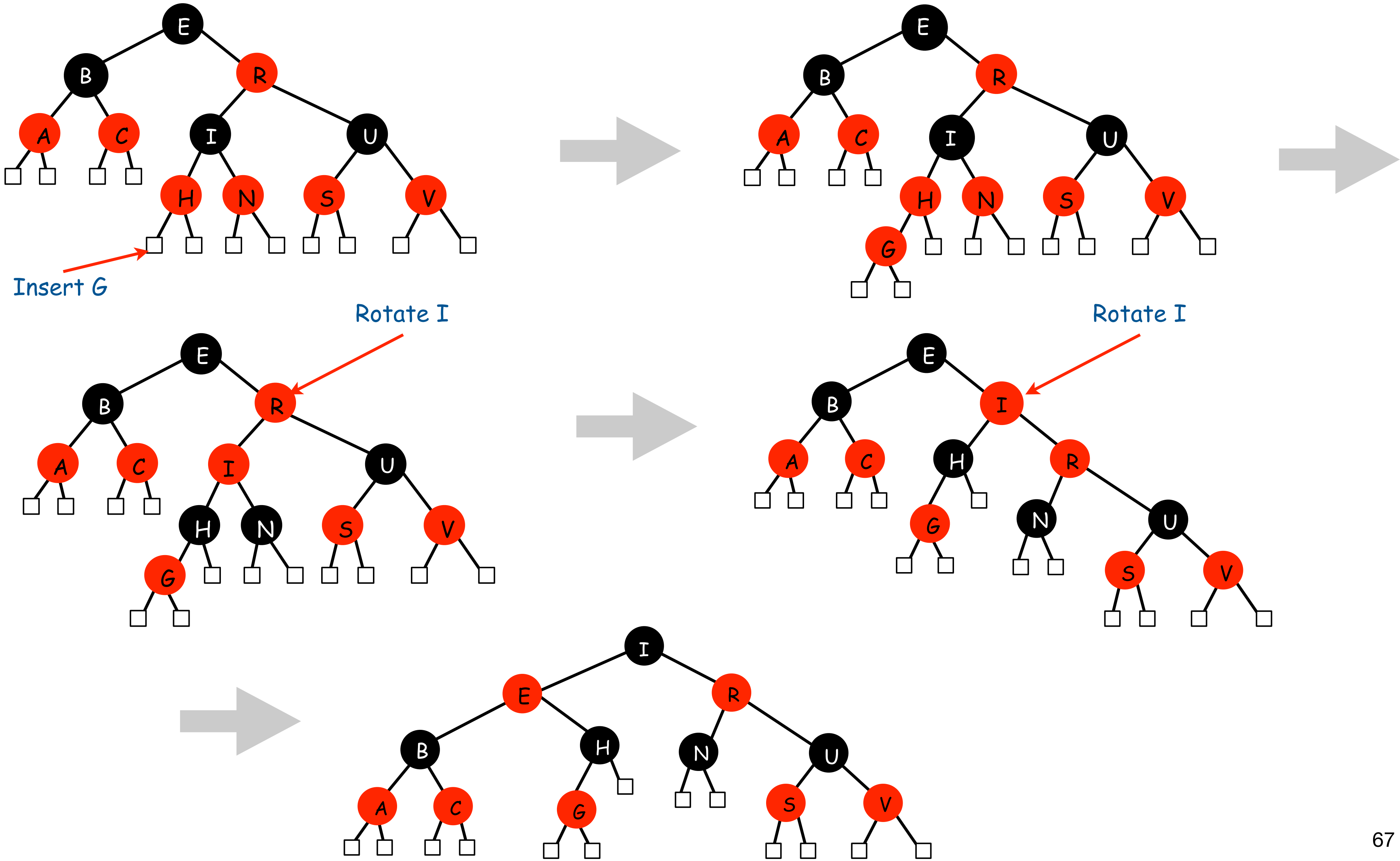
- Search to bottom after key (x)
- Insert red leaf
- Balance: 3 cases (+ symmetric)



Eksempel



Example



Running times in red-black trees

- Time for insertion:
  - Search to bottom after key: O(h)
  - Insert red leaf: O(1)
  - Perform recoloring and rotations on way up: O(h)
    - Can recolor many times (but at most h)
    - At most 2 rotations.
- Total O(h).

- Time for search
  - Same as BST: O(h)

- Height: O(log n)

Dynamic set implementations

Worst case running times

Implementation	search	insert	delete	minimum	maximum	successor	predecessor
linked lists	O(n)	O(1)	O(1)	O(n)	O(n)	O(n)	O(n)
ordered array	O(log n)	O(n)	O(n)	O(1)	O(1)	O(log n)	O(log n)
BST	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)	O(h)
2-3-4 tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)
red-black tree	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)	O(log n)

Balanced trees: implementations

Redblack trees:

Java: **`java.util.TreeMap, java.util.TreeSet.`**

C++ STL: map, multimap, multiset.

Linux kernel: **`linux/rbtree.h.`**