Dynamic Programming

CLRS chapter 15.0, 15.1, 15.4.

Dynamic Programming

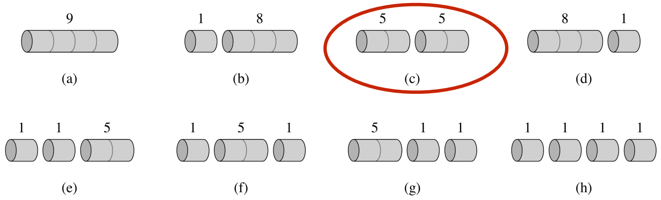
- General algorithmic technique
- Can be used when the problem have "optimal substructure":
  - *solution can be constructed from optimal solutions to subproblems.*
  - *use dynamic programming when subproblems overlap.*

- Today and next week: 4 examples
  - Rod cutting
  - Longest common subsequence
  - All pairs shortest path (next week)
  - Sequence alignment (next week)

Rod cutting

- Rod cutting problem: Cut a steel rod into pieces in order to maximize revenue. Cuts are free and each length has a revenue/price.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

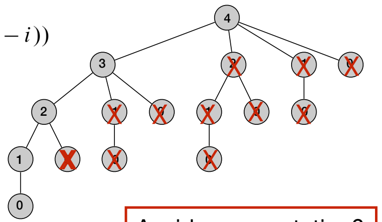


Rod cutting: Recursive solution

$$r_n = \begin{cases} 0 & \text{if } n = 0 \\ \max_{i=1\ldots n}(p_i + r_{n-i}) & \text{otherwise} \end{cases}$$

```
CUT-ROD(p, n)
if n == 0
    return 0
q = -∞
for i = 1 to n
    q = max(q, p[i] + CUT-ROD(p, n - i))
return q
```

time O(2^n)

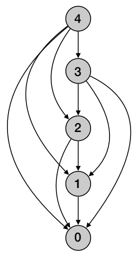


Avoid recomputation?

Rod cutting: Recursion + memoization

MEMOIZED-CUT-ROD(p, n)
let $r[0..n]$ be a new array
for $i = 0$ **to** n
    $r[i] = -\infty$
return MEMOIZED-CUT-ROD-AUX(p, n, r)

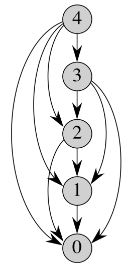
MEMOIZED-CUT-ROD-AUX(p, n, r)
if $r[n] \geq 0$
    return $r[n]$
if $n == 0$
    $q = 0$
else $q = -\infty$
    for $i = 1$ **to** n
        $q = \max(q, p[i] + $ MEMOIZED-CUT-ROD-AUX$(p, n - i, r))$
$r[n] = q$
return q



- Running time $\Theta(n^2)$:
  - each subproblem solved once,
  - time to solve a subproblem of size j: for loop iterates j times.
- Space: $\Theta(n)$

Rod cutting: bottom-up

BOTTOM-UP-CUT-ROD(p, n)
let $r[0..n]$ be a new array
$r[0] = 0$
for $j = 1$ **to** n
    $q = -\infty$
    for $i = 1$ **to** j
        $q = \max(q, p[i] + r[j - i])$
    $r[j] = q$
return $r[n]$



- running time $\Theta(n^2)$.
- Space $\Theta(n)$

Rod cutting: reconstruction solution

PRINT-CUT-ROD-SOLUTION(p, n)
$(r, s) = $ EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while $n > 0$
    print $s[n]$
    $n = n - s[n]$

EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let $r[0..n]$ and $s[0..n]$ be new arrays
$r[0] = 0$
for $j = 1$ **to** n
    $q = -\infty$
    for $i = 1$ **to** j
        if $q < p[i] + r[j - i]$
            $q = p[i] + r[j - i]$
            $s[j] = i$
    $r[j] = q$
return r and s

	r	s
1	1	1
2	5	2
3	8	3
4	10	2
5	13	2
6	17	6
7	18	1
8	22	2

Running time (print): $\Theta(n)$
Total running time: $\Theta(n^2)$
Space: $\Theta(n)$

Longest common subsequence

Longest common subsequence

- subsequence:

P I N E

S P R I N G T I M E

Longest common subsequence

- subsequence:

Longest common subsequence

- subsequence:

- Longest common subsequence

B A N A N A S

S A N D A L S

Longest common subsequence

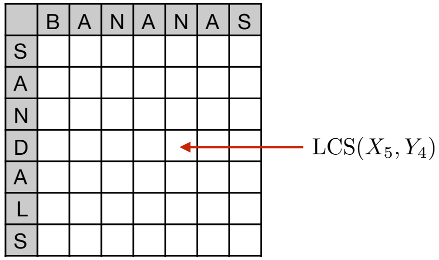
- subsequence:

- Longest common subsequence

Longest common subsequence

- Subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$



Longest common subsequence

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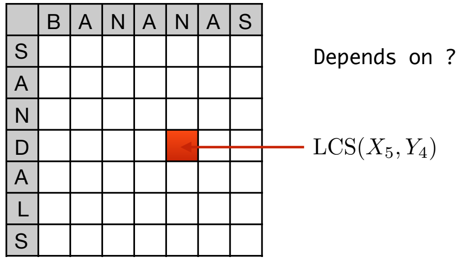

Longest common subsequence

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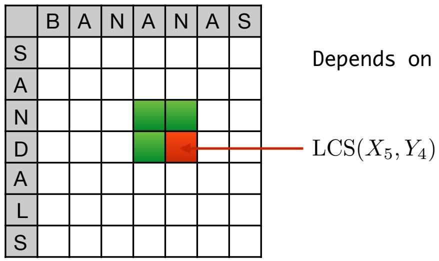

Longest common subsequence

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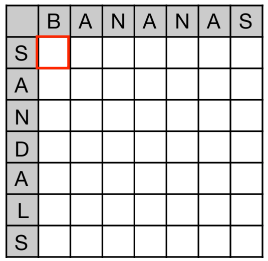

Longest common subsequence

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	B	A	N	A	N	A	S
S	0						
A							
N							
D							
A							
L							
S							

Longest common subsequence

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S	0						
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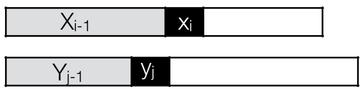

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S	0	0					
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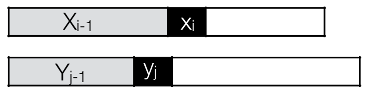

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S	0	0	0	0	0	0	
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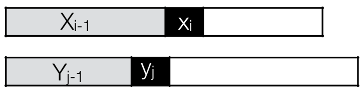

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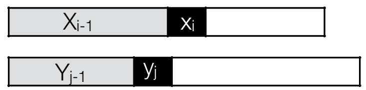

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S	0	0	0	0	0	0	1
A	0	1	1				
N							
D							
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L							
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Longest common subsequence

- subproblem property:

X_{i-1} | x_i

Y_{j-1} | y_j

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S	0	0	0	0	0	0	1
A	0	1	1	1			
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S	0	0	0	0	0	0	1
A	0	1	1	1			
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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1		
N							
D							
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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1		
N							
D							
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L							
S							

Longest common subsequence

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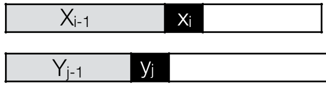

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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

Running time: Θ(nm)
Space: Θ(nm)

Linear space: save only last and current row

Longest common subsequence

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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖

Longest common subsequence

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S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖

Longest common subsequence

```
PRINT-LCS(b, X, i, j)
if i == 0 or j = 0
    return
if b[i, j] == "↖"
    PRINT-LCS(b, X, i − 1, j − 1)
    print x_i
elseif b[i, j] == "↑"
    PRINT-LCS(b, X, i − 1, j)
else PRINT-LCS(b, X, i, j − 1)
```

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖