Dynamic Programming

CLRS chapter 15.0, 15.1, 15.4.

Dynamic Programming

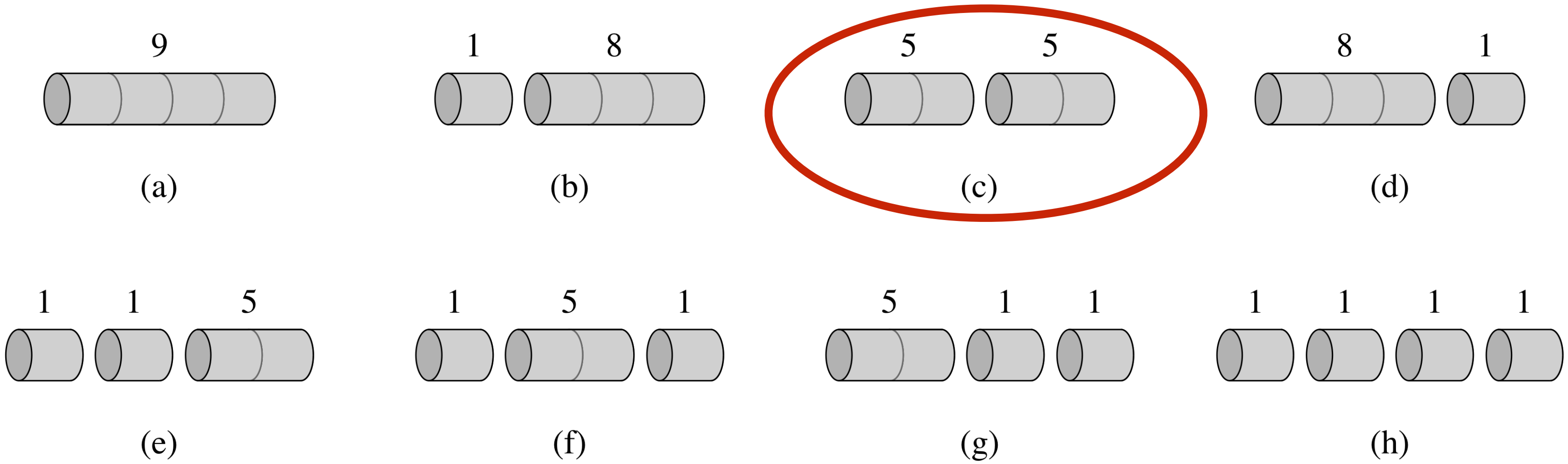
- General algorithmic technique
- Can be used when the problem have "optimal substructure":
  - ✦ *solution can be constructed from optimal solutions to subproblems.*
  - ✦ *use dynamic programming when subproblems overlap.*

- Today and next week: 4 examples
  - Rod cutting
  - Longest common subsequence
  - All pairs shortest path (next week)
  - Sequence alignment (next week)

Rod cutting

- Rod cutting problem: Cut a steel rod into pieces in order to maximize revenue. Cuts are free and each length has a revenue/price.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

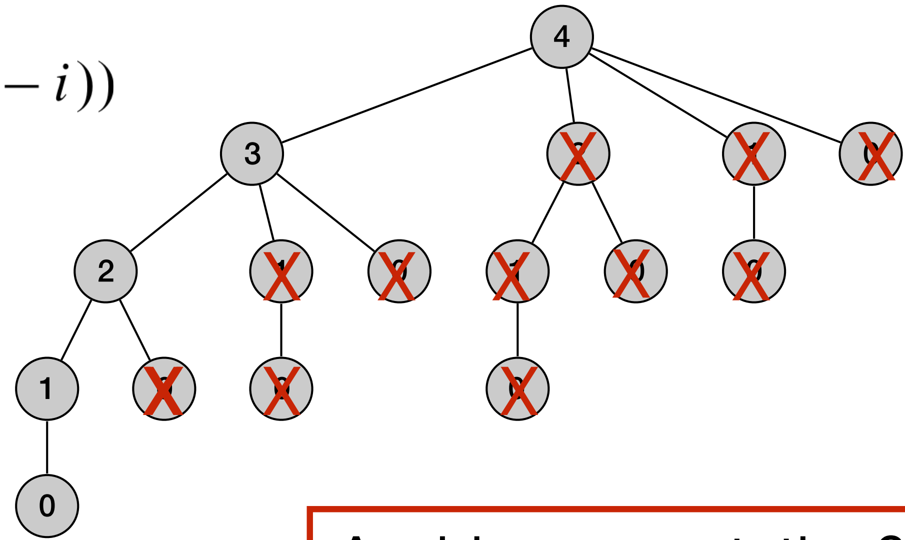


Rod cutting: Recursive solution

$$r_n = \begin{cases} 0 & \text{if } n = 0 \\ \max_{i=1\ldots n}(p_i + r_{n-i}) & \text{otherwise} \end{cases}$$

```
CUT-ROD(p, n)
if n == 0
    return 0
q = −∞
for i = 1 to n
    q = max(q, p[i] + CUT-ROD(p, n − i))
return q
```

time O(2^n)



Avoid recomputation?

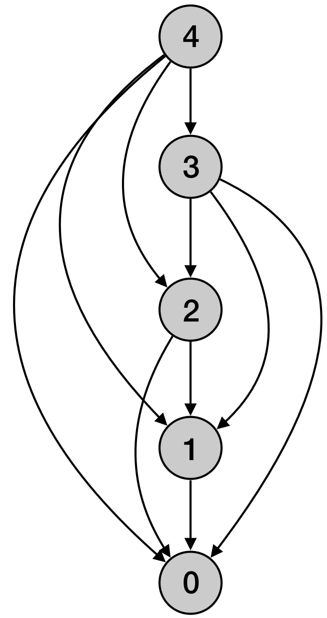
Rod cutting: Recursion + memoization

MEMOIZED-CUT-ROD(p, n)

let $r[0..n]$ be a new array
for $i = 0$ **to** n
    $r[i] = -\infty$
return MEMOIZED-CUT-ROD-AUX(p, n, r)

MEMOIZED-CUT-ROD-AUX(p, n, r)

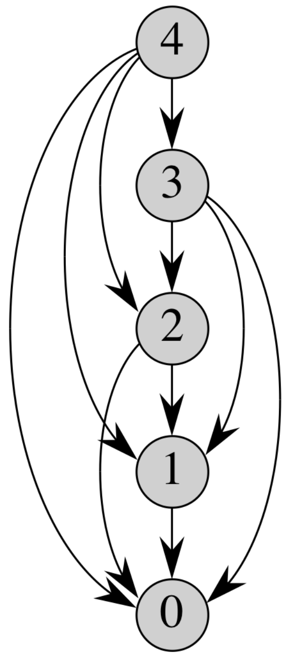
if $r[n] \geq 0$
    return $r[n]$
if $n == 0$
    $q = 0$
else $q = -\infty$
    for $i = 1$ **to** n
        $q = \max(q, p[i] + \text{MEMOIZED-CUT-ROD-AUX}(p, n-i, r))$
$r[n] = q$
return q



- Running time $\Theta(n^2)$:
  - each subproblem solved once,
  - time to solve a subproblem of size j: for loop iterates j times.
- Space: $\Theta(n)$

Rod cutting: bottom-up

```
BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = −∞
    for i = 1 to j
        q = max(q, p[i] + r[j − i])
    r[j] = q
return r[n]
```



- running time $\Theta(n^2)$.
- Space $\Theta(n)$

Rod cutting: reconstruction solution

```
PRINT-CUT-ROD-SOLUTION(p, n)
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i
    r[j] = q
return r and s
```

	r	s
1	1	1
2	5	2
3	8	3
4	10	2
5	13	2
6	17	6
7	18	1
8	22	2

Running time (print): $\Theta(n)$
Total running time: $\Theta(n^2)$
Space: $\Theta(n)$

Longest common subsequence

Longest common subsequence

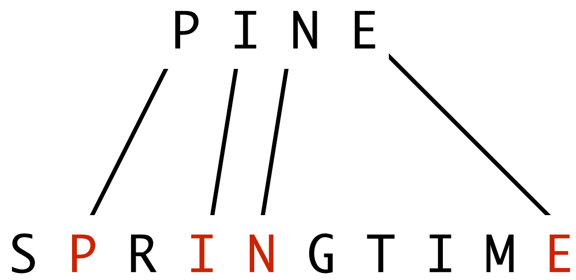
- subsequence:

```
P I N E

S P R I N G T I M E
```

Longest common subsequence

- subsequence:



Longest common subsequence

- subsequence:

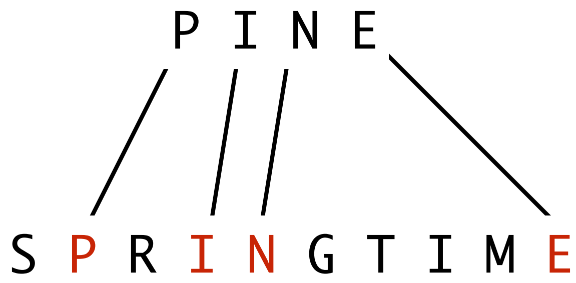
- Longest common subsequence

B A N A N A S

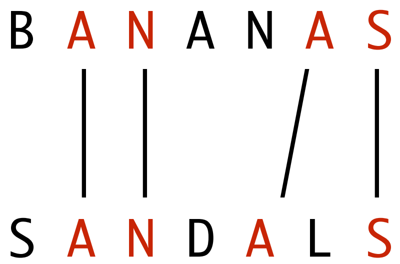
S A N D A L S

Longest common subsequence

- subsequence:



- Longest common subsequence



Longest common subsequence

- Subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S							
A							
N							
D							
A							
L							
S							

$\mathrm{LCS}(X_5, Y_4)$

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S							
A							
N							
D							
A							
L							
S							

Depends on ?

$\mathrm{LCS}(X_5, Y_4)$

Longest common subsequence

- subproblem property:

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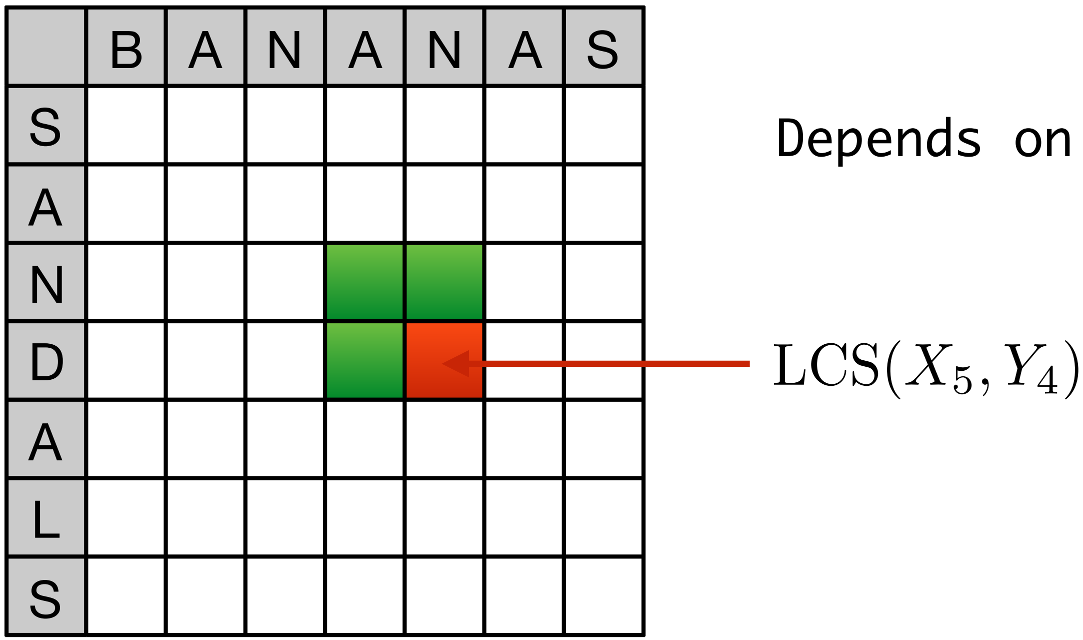

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S							
A							
N							
D							
A							
L							
S							

Longest common subsequence

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$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0						
A							
N							
D							
A							
L							
S							

Longest common subsequence

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	B	A	N	A	N	A	S
S	0						
A							
N							
D							
A							
L							
S							

Longest common subsequence

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	B	A	N	A	N	A	S
S	0	0					
A							
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	
A							
N							
D							
A							
L							
S							

Longest common subsequence

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$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A							
N							
D							
A							
L							
S							

Longest common subsequence

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$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A							
N							
D							
A							
L							
S							

Longest common subsequence

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$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0						
N							
D							
A							
L							
S							

Longest common subsequence

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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0						
N							
D							
A							
L							
S							

Longest common subsequence

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$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1					
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1					
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1				
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1				
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1			
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

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	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1			
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1		
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1		
N							
D							
A							
L							
S							

Longest common subsequence

- subproblem property:

$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	
N							
D							
A							
L							
S							

Longest common subsequence

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$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

Running time: Θ(nm)
Space: Θ(nm)

Linear space: save only
last and current row

Longest common subsequence

- subproblem property:

$$\text{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \text{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\text{LCS}(X_i, Y_{j-1}), \text{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖

Longest common subsequence

- subproblem property:

$$\mathrm{LCS}(X_i, Y_j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \mathrm{LCS}(X_{i-1}, Y_{j-1}) + 1 & \text{if } x_i = y_j \\ \max(\mathrm{LCS}(X_i, Y_{j-1}), \mathrm{LCS}(X_{i-1}, Y_j)) & \text{if } x_i \neq y_j \end{cases}$$

	B	A	N	A	N	A	S
S	0	0	0	0	0	0	1
A	0	1	1	1	1	1	1
N	0	1	2	2	2	2	2
D	0	1	2	2	2	2	2
A	0	1	2	3	3	3	3
L	0	1	2	3	3	3	3
S	0	1	2	3	3	3	4

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖

Longest common subsequence

PRINT-LCS(b, X, i, j)

if i == 0 or $j = 0$
    return
if $b[i, j]$ == "↖"
    PRINT-LCS$(b, X, i-1, j-1)$
    print x_i
elseif $b[i, j]$ == "↑"
    PRINT-LCS$(b, X, i-1, j)$
else PRINT-LCS$(b, X, i, j-1)$

	B	A	N	A	N	A	S
S	↑	↑	↑	↑	↑	↑	↖
A	↑	↖	←	↖	←	↖	←
N	↑	↑	↖	←	↖	←	←
D	↑	↑	↑	↑	↑	↑	↑
A	↑	↖	↑	↖	←	↖	←
L	↑	↑	↑	↑	↑	↑	↑
S	↑	↑	↑	↑	↑	↑	↖