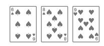
Randomized Algorithms

Randomized algorithms

- Today
  - Basic randomized algorithms
  - Expectation of random variables
    - Guessing cards
  - Selection
  - Quicksort

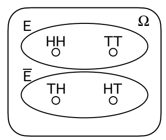


Random Variables and Expectation

Probability

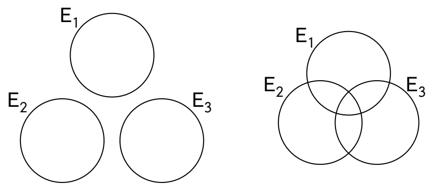
- Probability spaces.
  - Set of possible outcomes Ω.
  - Each item $i \in \Omega$ has probability $\Pr[i] \geq 0$ and $\sum_{i \in \Omega} \Pr[i] = 1$.
  - Event E is a subset of Ω and probability of E is $\Pr(E) = \sum_{i \in E} \Pr[i]$.
  - The complementary event $\overline{E}$ is $\Omega - E$ and $\Pr(\overline{E}) = 1 - \Pr(E)$.

- Example. Flip two fair coins.
  - $\Omega = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$.
  - $\Pr[i] = 1/4$ for each outcome i.
  - Event E = "the coins are the same"
  - $\Pr(\overline{E}) = 1/2$.



Probability

- Union bound.
  - What is the probability that any of event $E_1, \ldots, E_k$ will happen, i.e., what is $\Pr(E_1 \cup E_2 \cup \cdots \cup E_k)$?



  - If events are disjoint, $\Pr(E_1 \cup \cdots \cup E_k) = \Pr(E_1) + \cdots + \Pr(E_k)$.
  - If events overlap, $\Pr(E_1 \cup \cdots \cup E_k) < \Pr(E_1) + \cdots + \Pr(E_k)$.
  - In both cases, the union bound holds:

$$\Pr(E_1 \cup \cdots \cup E_k) \leq \Pr(E_1) + \cdots + \Pr(E_k)$$

Probability

- Independence.
  - Events E and F are independent if information about E does not affect outcome of F and vice versa.
  - Same as $\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$

Random variables

- A random variable is an entity that can assume different values.
- The values are selected “randomly”; i.e., the process is governed by a probability distribution.
- Examples: Let X be the random variable “number shown by dice”.
  - X can take the values 1, 2, 3, 4, 5, 6.
  - If it is a fair dice then the probability that X = 1 is 1/6:
    - Pr[X=1] = 1/6.
    - Pr[X=2] = 1/6.
    - …

Expected values

- Let X be a random variable with values in $\{x_1, \ldots x_n\}$, where x_i are numbers.
- The expected value (expectation) of X is defined as

$$E[X] = \sum_{j=1}^{n} x_j \cdot \Pr[X = x_j]$$

- The expectation is the theoretical average.
- Example:
  - X = random variable “number shown by dice”

$$E[X] = \sum_{j=1}^{6} j \cdot \Pr[X = j] = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} = 3.5$$

Waiting for a first succes

- Coin flips. Coin is heads with probability p and tails with probability $1-p$. How many independent flips X until first heads?
  - Probability of $X = j$? (first succes is in round j)

$$\Pr[X=j] = (1-p)^{j-1}\cdot p$$

  - Expected value of X:

$$E[X] = \sum_{j=1}^{\infty} j\cdot \Pr[X=j] \;=\; \sum_{j=1}^{\infty} j\cdot (1-p)^{j-1}\cdot p \;=\; \frac{p}{1-p}\sum_{j=1}^{\infty} j\cdot(1-p)^j$$

$$= \frac{p}{1-p}\cdot\frac{1-p}{p^2} = \frac{1}{p}$$

$$\sum_{k=0}^{\infty} k\cdot x^k = \frac{x}{(1-x)^2} \quad \text{for } |x|<1.$$

Properties of expectation

- If we repeatedly perform independent trials of an experiment, each of which succeeds with probability $p > 0$, then the expected number of trials we need to perform until the first succes is $1/p$.
- If X is a 0/1 random variable, then $E[X] = \Pr[X=1]$.
- Linearity of expectation: For two random variables X and Y we have

$$E[X+Y] = E[X]+E[Y]$$

Guessing cards

- Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.
- Memoryless guessing. Can't remember what's been turned over already. Guess a card from full deck uniformly at random.

- Claim. *The expected number of correct guesses is 1.*
  - $X_i = 1$ if i^{th} guess correct and zero otherwise.
  - $X =$ the correct number of guesses $= X_1 + \ldots + X_n$.
  - $E[X_i] = \Pr[X_i = 1] = 1/n$.
  - $E[X] = E[X_1+\cdots+X_n] = E[X_1]+\cdots+E[X_n] = 1/n+\cdots+1/n = 1.$

Guessing cards

- Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.
- Guessing with memory. Guess a card uniformly at random from cards not yet seen.
- Claim. *The expected number of correct guesses is $\Theta(\log n)$.*
  - $X_i = 1$ if i^{th} guess correct and zero otherwise.
  - $X =$ the correct number of guesses $= X_1 + \ldots + X_n$.
  - $E[X_i] = \Pr[X_i = 1] = 1/(n-i+1)$.
  - $E[X] = E[X_1]+\cdots+E[X_n] = 1/n+\cdots+1/2+1/1 = H_n$.

$$\ln n < H(n) < \ln n + 1$$

Coupon collector

- Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have at least 1 coupon of each type?
- Claim. *The expected number of steps is* $\Theta(n \log n)$.
  - Phase j = time between j and $j+1$ distinct coupons.
  - X_j = number of steps you spend in phase j.
  - X = number of steps in total = $X_0 + X_1 + \cdots + X_{n-1}$.
  - $E[X_j] = n/(n-j)$.
  - The expected number of steps:

$$E[X] = E[\sum_{j=0}^{n-1} X_j] = \sum_{j=0}^{n-1} E[X_j] = \sum_{j=0}^{n-1} n/(n-j) = n \cdot \sum_{i=1}^{n} 1/i = n \cdot H_n.$$

Median/Select

Select

- Given n numbers $S = \{a_1, \ldots, a_n\}$.
- Median: number that is in the middle position if in sorted order.
- Select(S,k): Return the kth smallest number in S.
  - Min(S) = Select(S,1), Max(S)= Select(S,n), Median = Select(S,$n/2$).
- Assume the numbers are distinct.

```
Select(S, k)

  Choose a pivot s ∈ S uniformly at random.

  For each element e in S:
   if e < s put e in S'
   if e > s put e in S''

  if |S'| = k-1 then return s

  if |S'| ≥ k then call Select(S', k)

  if |S'| < k then call Select(S'', k - |S'| - 1)
```

Select: Running time

```
Select(S, k)
  Choose a pivot s ∈ S uniformly at random.

  For each element e in S:
   if e < s put e in S'
   if e > s put e in S''

  if |S'| = k-1 then return s

  if |S'| ≥ k then call Select(S', k)

  if |S'| < k then call Select(S'', k - |S'| - 1)
```

- Worst case running time: $T(n) = cn + c(n-1) + c(n-2) + \ldots + c = \Theta(n^2)$
- If there is at least an ε fraction of elements both larger and smaller than s:

$$\begin{aligned} T(n) &= cn + (1-\varepsilon)cn + (1-\varepsilon)^2 cn + \ldots \\ &= (1 + (1-\varepsilon) + (1-\varepsilon)^2 + \ldots)cn \\ &\leq cn/\varepsilon \end{aligned}$$

- Intuition: A fairly large fraction of elements are “well-centered” => random pivot likely to be good.

Select: Analysis

- Central element: ≥ 1/4 of the elements in current S are smaller and ≥ 1/4 are larger.

S 

- If pivot central: size of set shrinks by at least a factor 3/4.
- At least half the elements are central ⇒ Pr[s is central] = 1/2.
- Phase j: Size of set at most $(3/4)^j n$ and at least $(3/4)^{j+1} n$.
  - Pivot central ⇒ current phase ends.
  - Expected number of iterations before a central pivot is found = 2.
- $X =$ number of steps taken by algorithm. $X_j =$ number of steps in phase j.
- Then $X = X_0 + X_1 + X_2 + \cdots$
- $E[X_j] = 2cn(3/4)^j$
- Expected running time:

$$E[X] = E[\sum_j X_j] = \sum_j E[X_j] = \sum_j 2cn\left(\frac{3}{4}\right)^j = 2cn\sum_j\left(\frac{3}{4}\right)^j \leq 8cn$$

Quicksort

Quicksort

- Given n numbers $S = \{a_1, \ldots, a_n\}$.
- Assume the numbers are distinct.

```
Quicksort(S)

   if |S| ≤ 1 return S

   else

    Choose a pivot s ∈ S uniformly at random.

    For each element e in S
      if e < s put e in S'
      if e > s put e in S''

   L = Quicksort(S')
   R = Quicksort(S'')

   return the sorted list L∘s∘R
```

Quicksort

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	19	3	1	7	8	17	9	2	13	23	10	25	24	16	21	11	14	12	15

Quicksort

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	19	3	1	7	8	17	9	2	13	23	10	25	24	16	21	11	14	12	15

Quicksort

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	19	3	1	7	8	17	9	2	13	23	6	25	24	16	21	11	14	12	15
5	3	1	7	8	2														
19	17	13	23	10	25	24	16	21	11	14	12	15							

Recurse!

Quicksort

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	19	3	1	7	8	17	9	2	13	23	6	25	24	16	21	11	14	12	15
1	2	3	5	7	8														
10	11	12	13	14	15	16	17	19	21	23	24	25							

Combine!

Quicksort

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	19	3	1	7	8	17	9	2	13	23	6	25	24	16	21	11	14	12	15
1	2	3	5	7	8														
10	11	12	13	14	15	16	17	19	21	23	24	25							
1	2	3	5	7	8	9	10	11	12	13	14	15	16	17	19	21	23	24	25

Quicksort: Analysis

- Worst case: $\Omega(n^2)$ comparisons.
- Best case: $O(n \log n)$
- Enumerate elements such that $a_1 \leq a_2 \leq \cdots \leq a_n$.
- Indicator random variable for all pairs $i < j$:

$$X_{ij} = \begin{cases} 1 & \text{if } a_i \text{ and } a_j \text{ are compared by the algorithm} \\ 0 & \text{otherwise} \end{cases}$$

- X total number of comparisons:

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

- Expected number of comparisons:

$$E[X] = E[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

Quicksort: Analysis

- Compute expected number of comparisons.
- Since X_{ij} is an indicator variable: $E[X_{ij}] \; = \; \Pr[X_{ij} = 1]$.
- a_i and a_j compared $\Leftrightarrow$ a_i or a_j is the first pivot element chosen from $Z_{ij} = \{a_i, \ldots, a_j\}$
- Pivot chosen independently uniformly at random $\Rightarrow$

  all elements from Z_{ij} equally likely to be chosen as first pivot from this set.
- We have $\Pr[X_{ij} = 1] \; = \; 2/(j - i + 1)$.
- Thus

$$E[X] \; = \; \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \; = \; \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[X_{ij} = 1] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k} \; \leq \; \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \; = \; 2 \sum_{i=1}^{n-1} H_n \; = \; 2n \cdot H_n \; \leq \; O(n \log n)$$