Randomized Algorithms II

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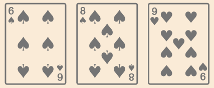
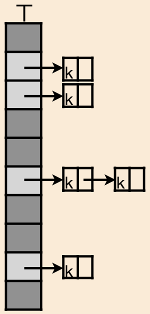
Thank you to Kevin Wayne and Philip Bille for inspiration to slides

Randomized algorithms

- Last weeks
  - Contention resolution
  - Global minimum cut
  - Expectation of random variables
    - Guessing cards
  - Quicksort
  - Selection



- Today
  - Hash functions and hash tables



Hashing

Dictionaries

- Dictionary problem. Maintain a dynamic set of $S \subseteq U$ subject to the following operations:
  - Lookup(x): return true if $x \in S$ and false otherwise
  - Insert(x): Set $S = S \cup \{x\}$
  - Delete(x): Set $S = S \setminus \{x\}$

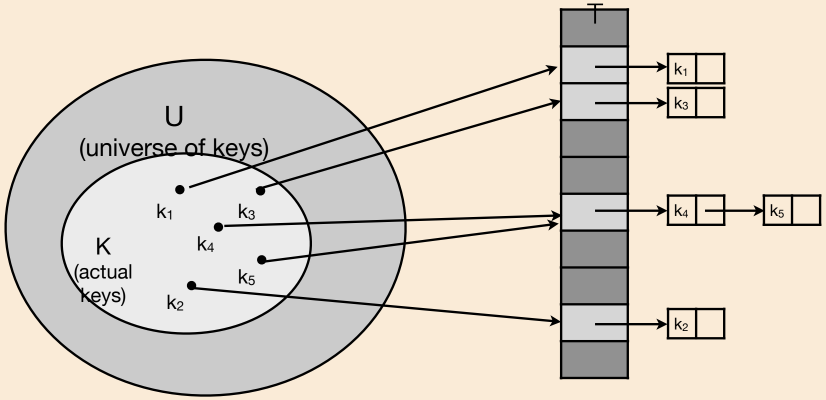
- Universe size. Typically $|U| = 2^{64}$ and $|S| << |U|$.
- Satellite information. Information associated with each element.

- Goal. A compact data structure with fast operations.

- Applications. Many! A key component in other data structures and algorithms.

Chained Hashing

- Chained hashing [Dumey 1956].
  - n = |S|.
  - Hash function. Pick some crazy, chaotic, random function h that maps U to {0, …, m-1}, where m = Θ(n).
  - Initialise an array A[0, …, m-1].
  - A[i] stores a linked list containing the keys in S whose hash value is i.



Uniform random hash functions

- *E.g.* $h(x) = x \bmod 11$. Not crazy, chaotic, random.
- Suppose $|U| \geq n^2$: For any hash function h there will be a set S of n elements that all map to the same position!

  => we end up with a single linked list.
- Solution: randomization.
  - For every element $u \in U$: select $h(u)$ uniformly at random in $\{0, \ldots, m-1\}$ independently from all other choices.
- Claim. *The probability that $h(u) = h(v)$ for two elements $u \neq v$ is $1/m$.*
- Proof.
  - m^2 possible choices for the pair of values $(h(u),h(v))$. All equally likely.
  - Exactly m of these gives a collision.

Chained Hashing with Random Hash Function

- Expected length of the linked list for h(x)?
- Random variable L_x = length of linked list for x. $L_x = |\{y \in S \mid h(y) = h(x)\}|$
- Indikator random variable:

$$I_y = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases} \qquad L_x = \sum_{y \in S} I_y \qquad E[I_y] = \Pr[h(y) = h(x)] = \frac{1}{m} \text{ for } x \neq y.$$

- The expected length of the linked list for x:

$$E[L_x] = E\left[\sum_{y \in S} I_y\right] = \sum_{y \in S} E[I_y] = 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} = 1 + (n-1) \cdot \frac{1}{m} = \Theta(1)\,.$$

Chained Hashing with Random Hash Function

- Constant time and O(n) space for the hash table.
- But:
    - Need O(|U|) space for the hash function.
    - Need a lot of random bits to generate the hash function.
    - Need a lot of time to generate the hash function.
- Do we need a truly random hash function?
- When did we use the fact that h was random in our analysis?

Chained Hashing with Random Hash Function

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Universal hash functions

- Universal hashing [Carter and Wegman 1979].
  - Let H be a family of functions mapping U to the set $\{0, \ldots, m-1\}$.
  - H is universal if for any $x, y \in U$, where $x \neq y$, and h chosen uniformly at random in H,

$$\Pr[h(x) = h(y)] \leq 1/m \, .$$

- Require that any $h \in H$ can be represented compactly and that we can compute the value $h(u)$ efficiently for any $u \in U$.

Universal Hashing

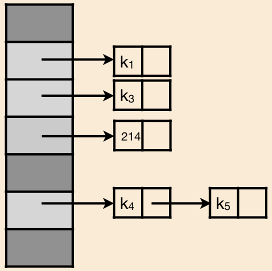
- Positional number systems. For integers x and b, the base-b representation of x is x written in base b.
- Example.
  - $(10)_{10} = (1010)_2$ $(1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0)$
  - $(107)_{10} = (212)_7$ $(2 \cdot 7^2 + 1 \cdot 7^1 + 2 \cdot 7^0)$

Universal Hashing

- **Hash function.** Given a prime p and a = $(a_1 a_2 \ldots a_r)_p$, define

$$h_a((x_1 x_2 \ldots x_r)_p) = a_1 x_1 + a_2 x_2 + \ldots + a_r x_r \mod p$$

- **Example.**
  - p = 7
  - a = $(107)_{10}$ = $(212)_7$
  - x = $(214)_{10}$ = $(424)_7$
  - $h_a(x) = 2 \cdot 4 + 1 \cdot 2 + 2 \cdot 4 \bmod 7 = 18 \bmod 7 = 4$



- **Universal family.**
  - $H = \{h_a \mid (a_1 a_2 \ldots a_r)_p \in \{0,\ldots,p-1\}^r\}$
  - Choose random hash function from H ~ choose random a.
  - H is universal (analysis next).
  - O(1) time evaluation.
  - O(1) space.
  - Fast construction.

Uniform Hashing

- **Lemma 1.** For any prime p, any integer $z \neq 0 \mod p$, and any two integers α, β:

$$\alpha z = \beta z \mod p \quad \Rightarrow \quad \alpha = \beta \mod p.$$

- **Proof.**
  - Show $(\alpha - \beta)$ is divisible by p:
    - $\alpha z = \beta z \mod p \quad \Rightarrow \quad (\alpha - \beta) z = 0 \mod p.$
    - By assumption z not divisible by p.
    - Since p is prime $\alpha - \beta$ must be divisible by p.
  - Thus $\alpha = \beta \mod p$ as claimed.

Universal Hashing

- Goal. For random $a = (a_1 a_2 \ldots a_r)_p$, show that if $x \neq y$ then $\Pr[h_a(x) = h_a(y)] \leq 1/p$.
  - Recall: $x = (x_1 x_2 \ldots x_r)_p$ and $y = (y_1 y_2 \ldots y_r)_p$:

$$x \neq y \Leftrightarrow (x_1 x_2 \ldots x_r)_p \neq (y_1 y_2 \ldots y_r)_p \Rightarrow x_j \neq y_j \text{ for some } j.$$

- Lemma 2. Let j be such that $x_j \neq y_j$. Assume the coordinates a_i have been chosen for all $i \neq j$. The probability of choosing a_j such that $h_a(x) = h_a(y)$ is $1/p$.
  - $h_a(x) = h_a(y) \quad \Leftrightarrow \quad \sum_{i=1}^{r} a_i x_i \mod p = \sum_{i=1}^{r} a_i y_i \mod p \quad \Leftrightarrow \quad a_j \boxed{(x_j - y_j)} = \boxed{\sum_{i \neq j} a_i (x_i - y_i)} \mod p$

fixed value z ≠ 0

fixed value since all a_i fixed for i≠j.

= c

  - *There is exactly one value $0 \leq a_j < p$ that satisfies $a_j z = c \mod p$.*
    - Assume there was two such values a_j and a'_j.
      - Then $a_j z = a'_j z \mod p$.
      - Lemma 1 $\Rightarrow a_j = a'_j \mod p$. Since $a_j < p$ and $a'_j < p$ we have $a_j = a'_j$.
  - Probability of choosing a_j such that $h_a(x) = h_a(y)$ is $1/p$.

Universal Hashing

- **Lemma 2.** Let j be such that $x_j \neq y_j$. Assume the coordinates a_i have been chosen for all $i \neq j$. The probability of choosing a_j such that $h_a(x) = h_a(y)$ is $1/p$.

- **Theorem.** For random $a = (a_1 a_2 \ldots a_r)_p$, if $x \neq y$ then

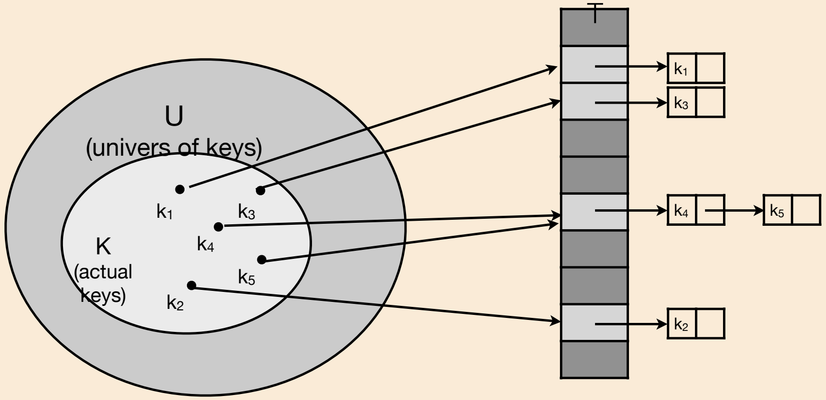
$$\Pr[h_a(x) = h_a(y)] = 1/p\,.$$

- **Proof.**
  - E : the event that $h_a(x) = h_a(y)$.
  - F_b : the event that the values a_i for $i \neq j$ gets the sequence of values b.
  - Lemma 2 shows that $\Pr[E \mid F_b] = 1/p$ for all b.
  - Thus

$$\Pr[E] = \sum_b \Pr[E \mid F_b] \cdot \Pr[F_b] = \sum_b \frac{1}{p} \cdot \Pr[F_b] = \; = \frac{1}{p} \sum_b \cdot \Pr[F_b] = \frac{1}{\mathrm{p}}$$

Dictionaries

- Theorem. We can solve the dictionary problem (without special assumptions) in:
  - O(n) space.
  - O(1) expected time per operation (lookup, insert, delete).



Universal Hashing

- Other universal families.
  - For prime p > 0.

$$h_{a,b}(x) = ax \mod p$$

$$H = \{h_{a,b} \mid a \in \{1,\ldots,p-1\},\, b \in \{0,\ldots,p-1\}\}\ .$$

  - Hash function from k-bit numbers to l-bit numbers.

$$h_a(x) = (ax \mod 2^k) \gg (k - l)$$

$$H = \{h_a \mid a \text{ is an odd integer in } \{1,\ldots,2^k - 1\}\}$$