Hashing

- Dictionaries
- Hashing with chaining
- Hash functions
- Linear Probing

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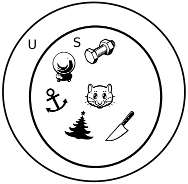
Dictionaries

Dictionary: Maintain a dynamic set S. Every element x has a key x.key from a universe U, along with satellite data x.data.

Operations:

- search(k) determine whether an element x with $x.\text{key} = k$ exists, and return it.
- insert(x) add x to the set S.
- delete(x) remove x from the set S.



insert()

search()

search()

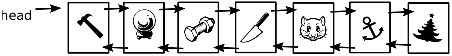
Dictionaries

Applications

- Basic data structure for representing a set
- Used in many algorithms and data structures

Challenge How can we solve the dictionary problem using current techniques?

Dictionaries - solution with a chained list - too slow!



Time:

search(k) - $O(|S|)$ time (search through all elements)

insert(x) - $O(1)$ time (insert at head of list)

delete(x) - $O(1)$ time to change pointers.

Space: $O(|S|)$ space.

Dictionaries - solution with an array - too large!

- A is an array of size U
- Save x on the position $A[x.key]$ in A.

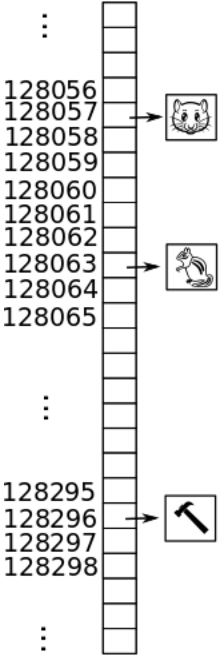
search(k) - $O(1)$ time to return $A[k]$

insert(x) - $O(1)$ time to set $A[x.\text{key}] = x$

delete(x) - $O(1)$ time to set $A[x.\text{key}] = \textit{null}$.

Space: $O(|U|)$

Exercise: When is this a problem?



Dictionaries - two dissatisfactory solutions

Data structure	`Search`	`Insert`	`Delete`	Space
Chained list	$O(\|S\|)$	$O(1)$	$O(1)$	$O(\|S\|)$
Array	$O(1)$	$O(1)$	$O(1)$	$O(\|U\|)$

Challenge: Can we do better?

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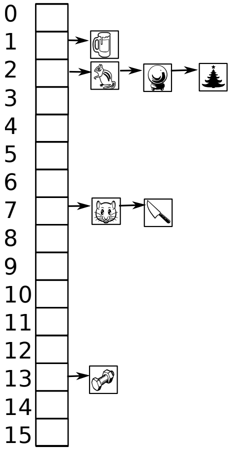
Hashing with chaining

Idea: Use a hash function $h : U \to \{0, \ldots, m\}$ where $m = O(|S|)$.

- Maintain an array A of size m,
- Each entry of the array points to a chained list,
- The element x is stored somewhere in the chained list at $A\,[h(x.key)]$.

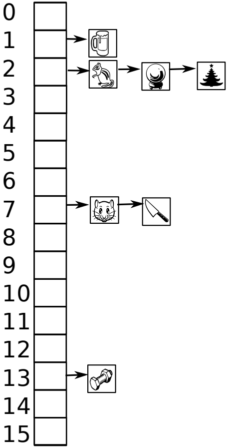
Collision: When $m < |U|$, then even when $x.key \neq y.key$, we risk $h(x.key) = h(y.key)$. We call this a *collision*.
We want h such that there are few collisions.
Hash (vb tr) "to confuse, muddle, or mess up".



Hashing with chaining

How it works.

- insert()
  h() = 0
- insert()
  h() = 7
- search()
  h() = 15
- search()
  h() = 2



Hashing with chaining

How it works.

search(k) - search through $A[k]$'s list for k.

insert(x) - insert x in $A[h(x.key)]$'s list.

delete(x) - delete x from list.

Time:

search(k) - $O(|\text{list's length}|)$ time

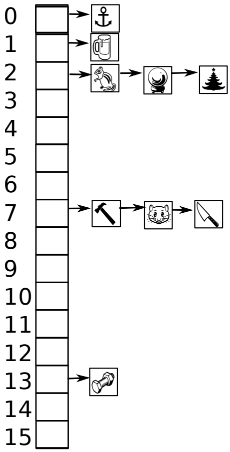
insert(x) - $O(1)$ time (at head of list)

delete(x) - $O(1)$ time to change pointers.

Plus the time it takes to calculate $h(x.key)$

Space:

$O(m + |S|) = O(|S|)$



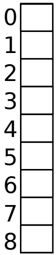
Hashing with chaining - Exercise

Insert the following keys K in a hash table of size 9 using hashing with chaining using the hash function

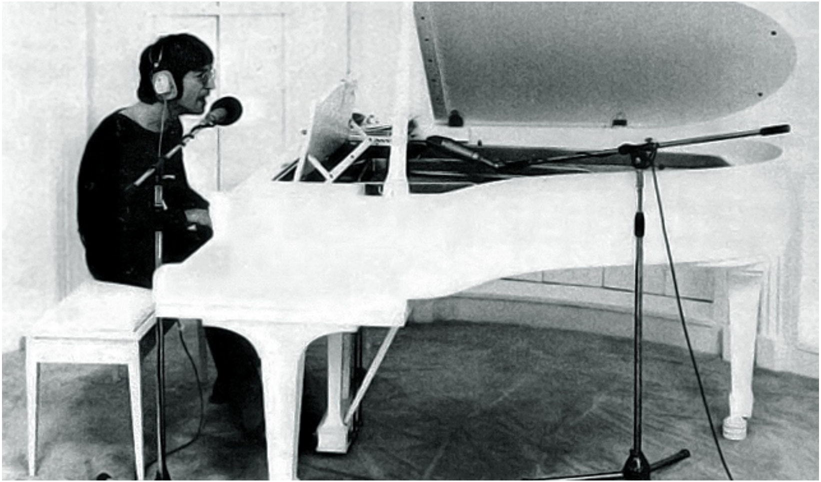
$$h(k) = k \mod 9$$

$K = 5, 28, 19, 15, 20, 33, 12, 17, 10$

How long is the longest list?



Uniform hashing



Imagine there's a
uniform hash function $h : U \to \{0, \dots, m-1\}$.

Uniform hashing

Definition (Load factor)

$\alpha = |S|/m$. The average length of lists.

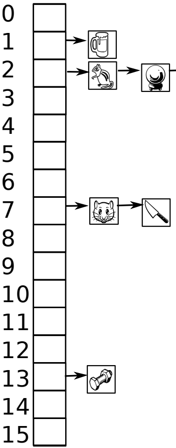
$m = \Theta(|S|) \Rightarrow \alpha = \Theta(1)$.

Dream world: Imagine there's a hash function h that is

- computable in $O(1)$ time, and
- For any $x \in U$: $h(x)$ is independent uniformly random in $\{0, \ldots, m-1\}$.

Then:

- Expected length of list $= \alpha$.
- $\Rightarrow$ search(k) in $O(\alpha) = O(1)$ time.
- Search, Insert, Delete: $O(1)$ time.
  $O(|S|)$ space.



Dictionaries - two dissatisfactory and one imaginary

Data structure	`Search`	`Insert`	`Delete`	Space
Chained list	$O(\|S\|)$	$O(1)$	$O(1)$	$O(\|S\|)$
Array	$O(1)$	$O(1)$	$O(1)$	$O(\|U\|)$
Hashing with chaining	$O(1)^\dagger$	$O(1)$	$O(1)$	$O(\|S\|)$

†: Expected running time. Assuming uniform hashing.

Challenge: Find a real-life hash function that works.

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Universal hashing

Goal: Avoid collisions $h(y) = h(x)$ for $x \neq y$.
If $h(x)$ and $h(y)$ are independent uniform random, then

$$\Pr[h(x) = h(y)] = 1/m$$

Definition (Universal hashfunction)

h is universal if for any $x, y \in U$ with $x \neq y$,
$\Pr[h(x) = h(y)] \leq 1/m$

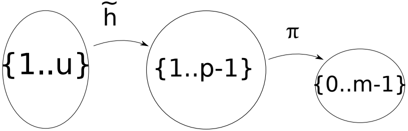
If h is universal, what is the expected size of the list at $A[h(x)]$?

$$\sum_{y \in S} \Pr[h(y) = h(x)] \leq 1 + \sum_{y \in S \setminus \{x\}} \frac{1}{m} \leq 1 + \frac{|S|}{m} = O(1)$$

All operations in (expected) $O(1)$ time!

Hash function: multiply-mod-prime

p is a prime $> |U|$.

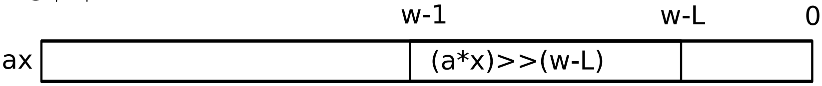


$$h_{a,b}(x) = ((ax + b) \mod p) \mod m$$

- Select $a \in \{1, \ldots, m-1\}$ and $b \in \{0, \ldots, m-1\}$ independently uniformly at random
- Use $h(x) = h_{a,b}(x) = \pi(\widetilde{h}_{a,b}(x))$ as hash function.
- $\widetilde{h}_{a,b}$ is collision free because $a \neq 0$
- π introduces collisions when $m < p$
- Given $x \neq y$, then $\Pr[h(x) = h(y)] < \frac{1}{m}$

Hash function: multiply-shift

Assume $|U|$ and m are powers of 2.
E.g $|U| = 2^w = 2^{64}$ and $m = 2^L$.



- Select odd $a \in \{1, 3, 5, \ldots, |U| - 1\}$
- $h_a(x) = \lfloor (ax \mod 2^w)/2^{w-L} \rfloor$
- Implementation: `return (a*x)»(64-L);`
- $\Pr\left[h_a(x) = h_a(y)\right] \leq \frac{2}{m}$

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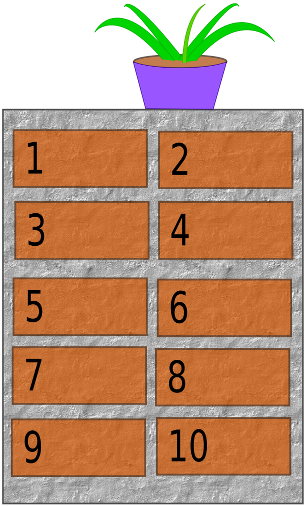
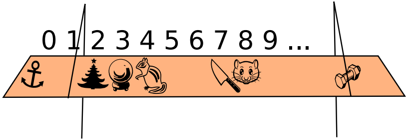
Analogies

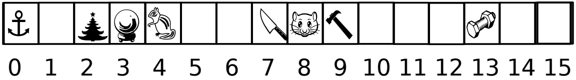


Chaining: Like a desk of drawers.
Must linear-search through
drawer no. 8 to find 🐹



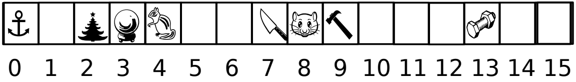
Linear Probing: Like a shelf.
No space for 🔨 at h(🔨) $= 7$,
So insert 🔨 at the nearest vacant
spot to the right.

Linear probing



- Maintain an array of size m
- Idea: Save x in $A[x.key]$
- Challenge: Collisions.

Linear probing

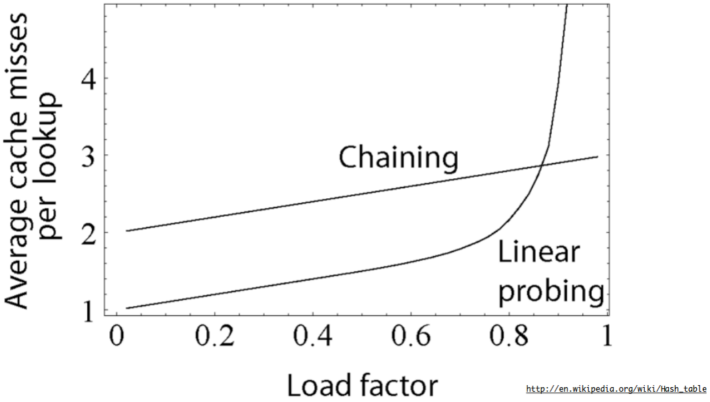


- Maintain an array of size m
- A *cluster* is a sequence of consecutive non-empty positions.
- Store x in $A[x.key]$, or somewhere in the cluster containing $x.key$, to the right of $x.key$.

Example:
Insert(). h() = 8.
Delete(). h() = 7. h()=7.
Search(). h() = 2.
Space: $m = O(|S|)$. Time: $O(|\text{cluster}|)$. $\leftarrow O(1)$ for some hash functions h.

Linear Probing

Huge advantage: Linear Probing is cache efficient.



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