Weekplan: Union-Find

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Reading

Introduction to Algorithms, Cormen, Rivest, Leisersons and Stein (CLRS): Chapter 21.1-21.4 and *Algorithms*, 4ed., Sedgewick and Wayne: Chapter 1.5.

Exercises

1 Run Union-Find by Hand Look at the following sequence of operations: $\text{INIT}(7)$, $\text{UNION}(3,4)$, $\text{UNION}(5,0)$, $\text{UNION}(4,5)$, $\text{UNION}(4,3)$, $\text{UNION}(0,1)$, $\text{UNION}(2,6)$, $\text{UNION}(0,4)$ and $\text{UNION}(6,0)$.

1.1 [w] Run the sequence of operations using fast find by hand. Show the contents of the *id* array after every step. Assume the $\text{UNION}(i,j)$ operation always updates *id* for the set given by i.

1.2 [w] Run the sequence using fast union by hand. Show the trees after every step. Assume $\text{UNION}(i,j)$ always sets the root of the tree given by i to be a child of the root of the tree given by j.

1.3 Run the sequence using weighted union by hand. Show the trees after very step. Assume $\text{UNION}(i,j)$ sets the root of the tree given by i to be a child of the root of the tree given by j when the sizes of two trees are equal.

1.4 Show the result of path compression after a $\text{FIND}(x)$ operation, where x is respectively a leaf, an internal node of depth 1, and an internal node of height 1, in one of the trees from the above exercises.

1.5 Give a sequence of operations that results in a tree of maximal depth using fast union.

1.6 Give a sequence of operations that results in a tree of maximal depth using weighted union.

1.7 Write pseudo code for a algorithm to do path compression. *Hint:* traverse the path twice.

2 Alternative to the Fast Find Algorithm One of your fellow students suggests the following intuitive variant of UNION fast find. Does it work?

```
UNION(i, j)
if FIND(i) ≠ FIND(j) then
  for k = 0 to n−1 do
    if id[k] == id[i] then
      id[k] = id[j]
    end if
  end for
end if
```

3 Dynamic Connected Components and Graph Search Using graph search (DFS or BFS) we can find the connected components of a graph. Give a simple solution for dynamic connected component using graph search and compare the complexity with the solutions for union and find.

4 Implementation of Union-Find We will like to implement data structures for union and find that supports INIT, UNION, and FIND.

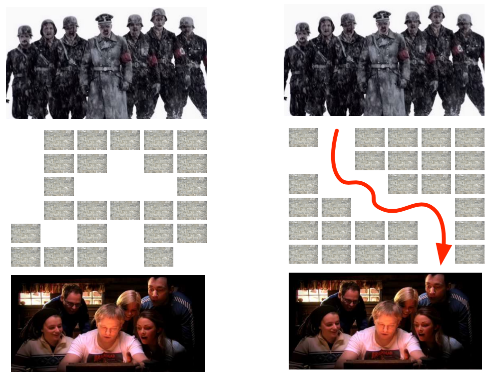
4.1 [BEng†] Implement fast find.

4.2 [†] Implement fast union.

4.3 [†] Extend the solution with weighted union.

4.4 [†] Extend the solution with path compression.

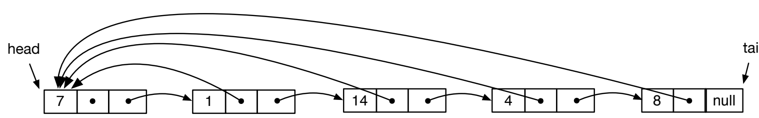
5 [$*$] **Zombie Invasion** In the post apocalyptic zombie world you and a small group of survivors have barricaded yourself in a small building. The only thing keeping the brutal zombies from eating you is a strong fortification. The fortification consists of a $k \times k$ grid of walls. Here illustrated by a 6×6 grid of walls.[1]



In the top of the grid the zombies are waiting to come in, and you and your group is located in the bottom. Unfortunately the walls are weak and collapse regularly. If a path of walls between the top and the bottom of the grid is collapsed the zombies can get in and eat you. In order to start evacuation you want to monitor if there is currently a path through the fortification (from top to bottom). Give a data structure that can efficiently keep track of this while the walls are collapsing one by one.

6 [$*$] **Recursive Path Compression** Write pseudo code for a *recursive* algorithm for path compression. *Hint:* it can be done with only few lines of code.

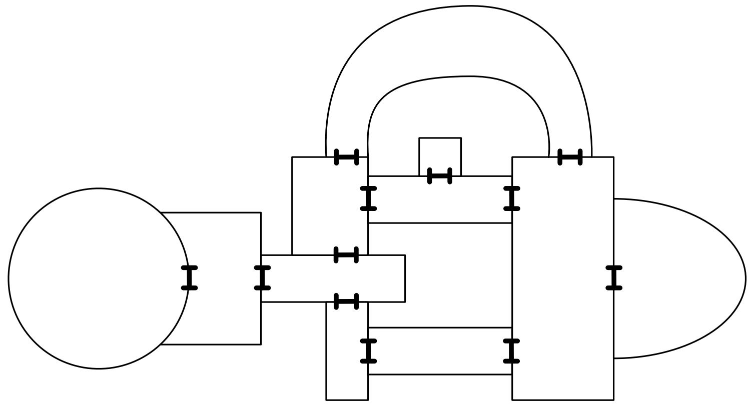
7 **Union-Find using Linked Lists and Weights** We want to implement a variant of fast find using linked lists in the following way. Each set is represented by a singly linked list. The representative for a set is the first element in the list and each element in the list has a pointer to the representative. Furthermore we maintain a pointer to the tail of the list. For instance the data structure for the set $\{1, 4, 7, 8, 14\}$ with representaive 7 could look like this:



- **7.1** Using the representation, show how to implement $\textsc{Init}(n)$ in $O(n)$ time, $\textsc{Find}(i)$ in $O(1)$ time and $\textsc{Union}(i, j)$ in $O(|S(i)|)$ time, where $S(i)$ is the set containing i.
- **7.2** Show how to extend the solution such that $\textsc{Init}$ and $\textsc{Find}$ runs in the same time, but the time for $\textsc{Union}(i, j)$ is $O(\min(|S(i)|, |S(j)|))$. *Hint:* maintain a little extra information.
- **7.3** [$*$] Show that for p $\textsc{Find}$ and m $\textsc{Union}$ operations on n elements the above solution gives the running time $O(p + m \log n)$.

M **Mandatory Exercise: Floor Plan (exercises from the exam 2015)** A *floor plan* consists of a set of R rooms $r_0, .., r_{R-1}$ and D doors $d_0, .., d_{D-1}$ that each connects exactly two rooms. Each room is a geometric figure and a door between two rooms are indicated by three small bold lines between the rooms. For instance the following floor plan P consists of 11 rooms and 12 doors.

[1] Pictures from "Død snø", 2009.



M.1 Describe how to model a floor plan as a graph.

M.2 Draw the graph corresponding to the floor plan P in the above example.

M.3 We are are interested in examining if it is possible to evacuate the rooms in case of fire. *The entrance* is a special room on the floor plan. A *fire door* is a door that will automatically close in case of fire. A room can be *evacuated* to the entrance if there is a connection from the room to the entrance that does not use any fire doors. Give an algorithm that given a floor plan, an entrance e and a set B of k fire doors determines if all rooms can be evacuated to e. Analyze the running time of your algorithm as a function of R, D, and k.

M.4 We are now interested in displaying art in the form of either a sculpture of a painting in all rooms in a nice fashion. A *route* on the floor plan is a sequence of rooms $r_0, \ldots, r_{z-1}$ such that room r_i and r_{i-1} are connected by a door, and $r_0 = r_{z-1}$. A room can be visited multiple times on a route. A route is *beautiful* if the rooms on the route alternates between having a sculpture and a painting. Look at the example floor plan P from before. Is it possible to display either a sculpture or a paiting in each room such that *all* routes that starts and ends in the room furthest to the left are beautiful?

M.5 Give an algorithm that given a floor plan and an entrance e determines if all routes that starts and ends in e are beautiful. We assume that there is a path from e to all rooms. Analyze the running time of your algorithm as a function of R and D.