Simplicial Complexes: Theory and Implementation

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DSC 2011 Workshop
Kgs. Lyngby, 25th August 2011

Affine independence

Definition (affine independce)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}$ be points in an n-dimensional Euclidean space E^n. We call them *affinely dependent* if

$$(\exists \mu_1, \ldots, \mu_{p+1} \in \mathbb{R}) \sum_{i=1}^{p+1} \mu_i = 1 \wedge \sum_{i=1}^{p+1} \mu_i \mathbf{v}_i = 0.$$

Otherwise, we call them *affinely independent*.

Examples:

- three non-colinear points in E^2 are affinely independent;
- four non-coplanar points in E^3 are affinely independent;

Simplex

Definition (Euclidean simplex)

Having $p+1$ affinely independent points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1} \in E^n$, an *Euclidean simplex* $\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{p+1} \rangle$ is a set of points given by a formula:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_{p+1} \mathbf{v}_{p+1},$$

where $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$ (σ is the *convex hull* of $\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}$).

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- σ is a closed set in E^n.

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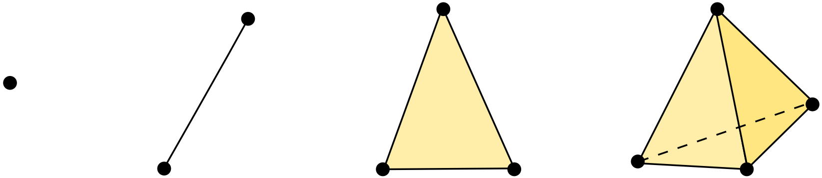
- σ is a closed set in E^n.
- p is the *dimension* of σ (equivalently σ is an Euclidean *p-simplex*).

Simplices



We call a 0-simplex a *vertex*, a 1-simplex an *edge*, a 2-simplex a *face* and a 3-simplex a *tetrahedron*.

Faces

Definition (vertex, *q*-face)

We call each point $\mathbf{v}_i$ a *vertex* of σ, and each simplex $\langle \mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_{q+1}} \rangle$ ($0 \leq q \leq p$, $1 \leq i_k \leq p+1$) a *q-face* of σ (or simply a *face* of σ, if no ambiguity arises).

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- The faces of σ that are not equal to σ itself are called its *proper faces*.

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- The faces of σ that are not equal to σ itself are called its *proper faces*.
- The union of all the boundary faces of a simplex σ is called the *boundary* of σ.

Simplex sets

- For arbitrary, finite set of simplices Σ we define its *dimension*, as the maximum dimension of the simplices in Σ:

$$\dim(\Sigma) = \max\{\dim(\sigma) : \sigma \in \Sigma\}.$$

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- We also define a *k-subset* of Σ as a set of all k-simplices in Σ:

$$\text{filter}_k(\Sigma) = \{\sigma_i \in \Sigma : \dim(\sigma_i) = k\}.$$

Euclidean simplicial complex

Definition (Euclidean simplicial complex)

A finite set Σ of Euclidean simplices forms a (finite) *Euclidean simplicial complex* if the following two conditions hold:

1. Σ is closed: for each simplex $\sigma \in \Sigma$, all faces of σ are also in Σ.
2. The intersection $\sigma_i \cap \sigma_j$ of any two simplices $\sigma_i, \sigma_j \in \Sigma$ is either empty or is a face of both σ_i and σ_j.

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- In particular, for any nonnegative integer k, the subset $\mathrm{K}^{(k)} \subset \mathrm{K}$ consisting of all simplices of dimension less than or equal to k is a subcomplex, called the *k-skeleton* of K.

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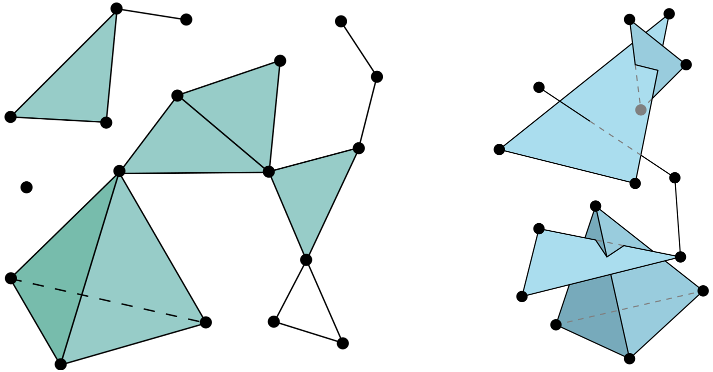
- The 0-skeleton of K is called a *vertex set* of K and denoted $V(\mathrm{K})$.

Euclidean simplicial complex



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- for $p < q$, the *coboundary relation* $C_{p,q}(\sigma^p)$ is the set of all q-simplices that have σ^p as a face:

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$$C_{p,q}(\sigma^p) = \mathrm{filter}_q\{\sigma \in \mathsf{K} : \mathrm{vert}(\sigma^p) \subset \mathrm{vert}(\sigma)\},$$

- for $p > 0$, the *adjacency relation* $A_p(\sigma^p)$ is the set of all p-simplices, which are $(p-1)$-adjacent to σ^p (which means those simplices, that share a $(p-1)$-face with σ^p):

$$A_p(\sigma^p) = \mathrm{filter}_p\{\sigma \in \mathsf{K} : |\mathrm{vert}(\sigma^p) \cap \mathrm{vert}(\sigma)| = p\},$$

Star

Definition (star)

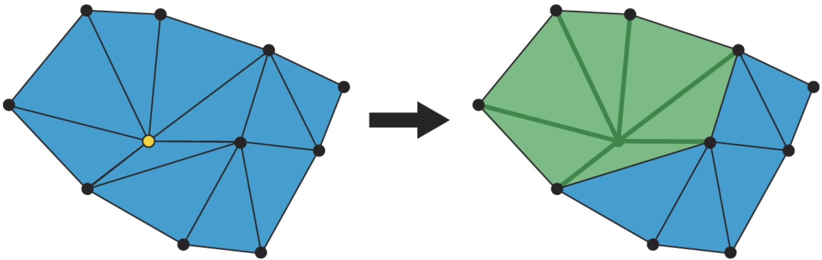
We define the *star* of a simplex σ as a set of all the simplices in K, which have σ as a face:

$$\mathrm{st}(\sigma^p) = \{\sigma \in \mathsf{K} : \mathrm{vert}(\sigma^p) \subset \mathrm{vert}(\sigma)\} = \bigcup_{q=p+1}^{n} C_{p,q}(\sigma^p).$$

For the sake of convenience, we also define a star of an arbitrary subset Σ of K, as the union of the stars of all simplices in Σ:

$$\mathrm{st}(\Sigma) = \bigcup_{\sigma_i \in \Sigma} \mathrm{st}(\sigma_i).$$

Star



Closure

Definition (closure)

We define the *closure* of a simplex $\sigma^p \in K$ as a set

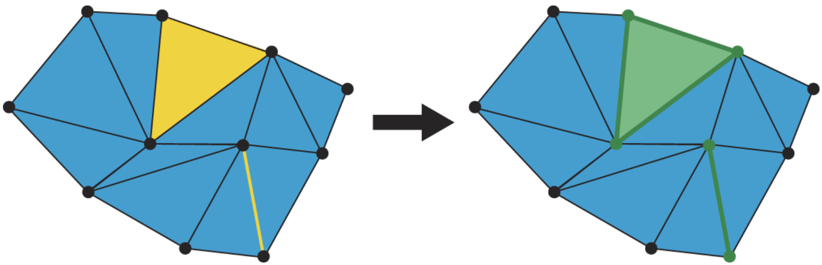
$$\mathrm{cl}(\sigma^p) = \bigcup_{q=0}^{p} B_{p,q}(\sigma^p).$$

The closure of a simplex set $\Sigma \subset K$ is expressed as a set

$$\mathrm{cl}(\Sigma) = \bigcup_{\sigma_i \in \Sigma} \mathrm{cl}(\sigma_i).$$

Equivalently, we can define the closure of a simplex $\sigma \in K$ (simplex set $\Sigma \subset K$) as the smallest subcomplex of K containing σ (including Σ).

Closure



Link

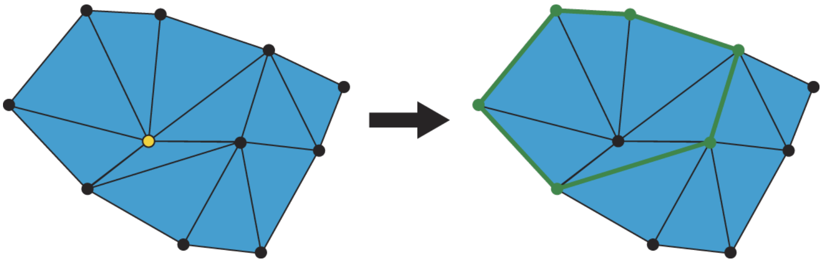
Definition (link)

The *link* of a simplex σ is defined as the set of all the the simplices in the closure of the star of σ, which do not share a face with σ:

$$\mathrm{lk}(\sigma) = \mathrm{cl}(\mathrm{st}(\sigma)) - \mathrm{st}(\mathrm{cl}(\sigma)).$$

It can be proven that for every simplex $\sigma \in$ K, $\mathrm{lk}(\sigma)$ is a subcomplex.

Link



Carrier

Definition

The *carrier* $\|K\|$ of a simplicial complex K (also called the *polyhedron* $\|K\|$) is a subset of E^n defined by the union, as point sets, of all the simplices in K.

Definition

For each point $v \in \|K\|$ there exists exactly one simplex $\sigma \in K$ containing v in its relative interior. This simplex is denoted by $\mathrm{supp}(v)$ and called the *support* of the point v.

Manifoldness

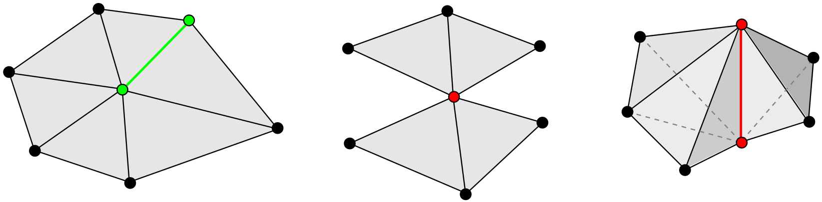
Definition (notion of manifoldness)

We say that a point $\mathbf{v} \in A \subset E^n$ is *p-manifold* if there exists a neighbourhood U of v such that $A \cap U$ is homeomorphic to $\mathbb{R}^p$ or $\mathbb{R}^{(p-1)} \times (0, +\infty)$. Otherwise we call $\mathbf{v}$ *non-manifold*.

We say that a simplex $\sigma \in$ K is p-manifold, if every point of the relative interior of this simplex is p-manifold with regard to the carrier of K. E.g. obviously each n-simplex is n-manifold, each $(n-1)$-simplex is n-manifold if it is a face of at least one n-simplex, etc.

We also say that an n-dimensional simplicial complex K in E^n is manifold, if each of its simplices is n-manifold.

Manifoldness



Orientations

We introduce the following equivalence relation in the set P_σ of all orderings $(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_{p+1}})$ of the vertices of $\sigma = \langle \mathbf{v}_1,\ldots,\mathbf{v}_{p+1}\rangle$:

$$(\mathbf{v}_{i_1},\ldots,\mathbf{v}_{i_{p+1}}) \sim (\mathbf{v}_{\pi(i_1)},\ldots,\mathbf{v}_{\pi(i_{p+1})})$$

iff $\pi : \{1,2,\ldots,p+1\} \longrightarrow \{1,2,\ldots,p+1\}$ is an even permutation operator.

We call each element of the quotient set $\mathscr{O}_\sigma = P_\sigma/\sim$ an *orientation* of a simplex σ.

If $p > 0$, $|\mathscr{O}_\sigma| = 2$, meaning that there are two possible orientations for any simplex defined on a set of $p+1$ points from E^n.

Oriented volume

We define an *oriented volume* of $\sigma = [\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}]$

$$\begin{aligned} \mathscr{V}(\sigma) &= \mathscr{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}) \\ &= \frac{1}{p!}\det(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_p - \mathbf{v}_{p+1}, \mathbf{v}_{p+1} - \mathbf{v}_1). \end{aligned}$$

It can be proven, that for an even permutation operator π

$$\mathscr{V}(\mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(p+1)}) = \mathscr{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}),$$

and for an odd permutation operator π'

$$\mathscr{V}(\mathbf{v}_{\pi'(1)}, \ldots, \mathbf{v}_{\pi'(p+1)}) = -\mathscr{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}),$$

Natural and induced orientation

Definition (natural orientation)

The orientation of σ, for which $\mathscr{V}(\sigma) > 0$ is called the *natural orientation*.

Definition (induced orientation)

The p-simplex $\sigma^p = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}]$ determines an orientation of each of its $(p-1)$-faces, called the *induced orientation*, by the following rule: the induced orientation on the face $\sigma_i^{p-1} = \langle \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{p+1} \rangle$ is defined to be $(-1)^{i+1} [\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_{p+1}]$.

Consistency

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- If $\sigma_i^n, \sigma_j^n \in K$ are two n-simplices that share an $(n-1)$-face σ^{n-1}, we say that orientations of σ_i^n and σ_j^n are *consistent* if they induce opposite orientations on σ^{n-1}.

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- An orientation of K is a choice of orientation of each n-simplex in such a way that any two simplices that interesect in an $(n-1)$-face are consistently oriented.
- If a complex K admits an orientation, it is said to be *orientable*.

Triangle meshes

Definition (triangle mesh)

A dimension 2 simplicial complex $\mathrm{K} \subset E^n$ (where $n \geq 2$), such that every 0 or 1-simplex $\sigma \in \mathrm{K}$ is a face of a 2-simplex $\sigma^2 \in \mathrm{K}$ is called a *triangle mesh*.

Triangle meshes inherit the notions of manifoldness and orientability from simplicial complexes.

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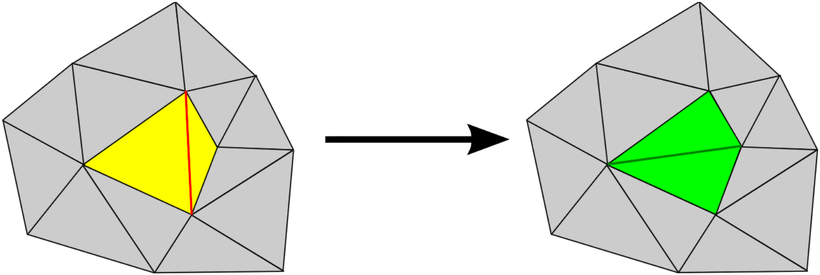
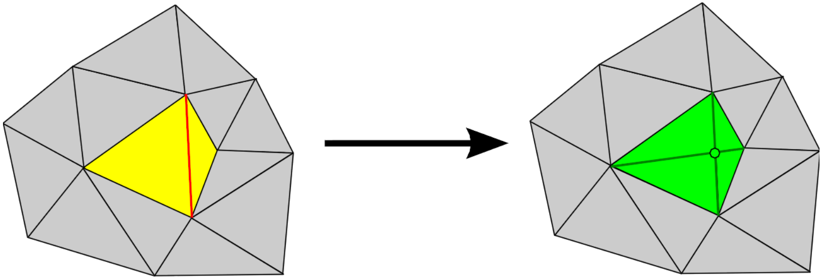
- **edge flips**: mesh reconnection without changing vertex placement;
- **edge splits**: introducing a new vertex on an edge;
- **face splits**: introducing a new vertex on a face;
- **edge collapse**: removing an edge and its adjacent triangles;

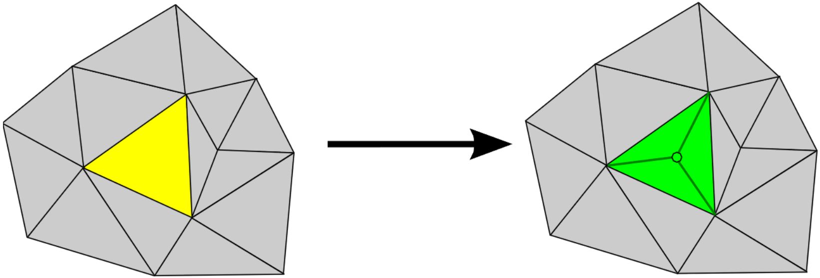
Edge flip



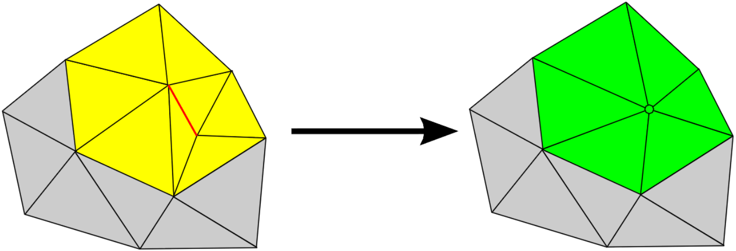
Edge split



Face split



Edge collapse



Tetrahedral meshes

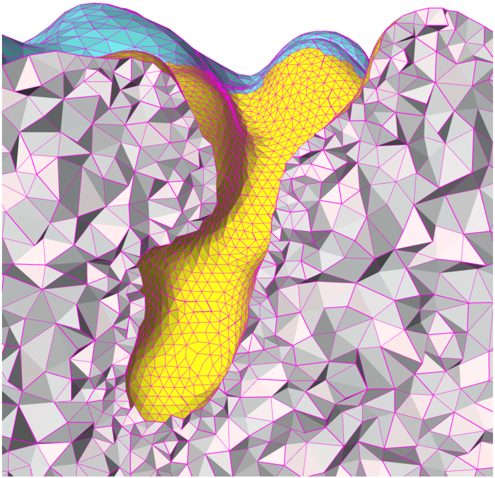
Definition (tetrahedral mesh)

A dimension 3 simplicial complex $K \subset E^n$ (where $n \geq 3$), such that every 0, 1 or 2-simplex $\sigma \in K$ is a face of a 3-simplex $\sigma^3 \in K$ is called a *tetrahedral mesh*.

Tetrahedral meshes inherit the notions of manifoldness and orientability from simplicial complexes.

Triangle mesh operations generalize (although not always easily) to tetrahedral meshes.

Tetrahedral mesh



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- If want to ensure efficient traversal, *incidence* information has to be stored together with the simplices.
- Examples include: quad-edge, half-edge (for 2-manifold triangular meshes).

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- Each simplex in K has its representation in IS data structure.
- We store with each p-simplex $\sigma^p \in$ K (for $p > 1$) the *unordered* set of handles to its $p+1$ $(p-1)$-dimensional faces $\sigma_1^{p-1}, \ldots, \sigma_{p+1}^{p-1}$ (the boundary relation $B_{p,p-1}(\sigma^p)$).

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- In order to make the traversal efficient, *partial coboundary relation* $C^*_{p,p+1}(\sigma^p)$ is also stored with every p-simplex $\sigma^p \in$ K, for $p < n$.
- Partial coboundary relation $C^*_{p,p+1}(\sigma^p)$ consists of $(p+1)$-simplices from $\mathrm{st}(\sigma^p)$ connecting σ^p with its link, one per each connected component in $\mathrm{lk}(\sigma^p)$.

Our implementation

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- Our implementation is orientation-aware: we identify an oriented simplex σ^p with an *ordered* tuple of its $(p-1)$-faces:

$$\left[\sigma_1^{p-1}, \ldots, \sigma_{p+1}^{p-1}\right],$$

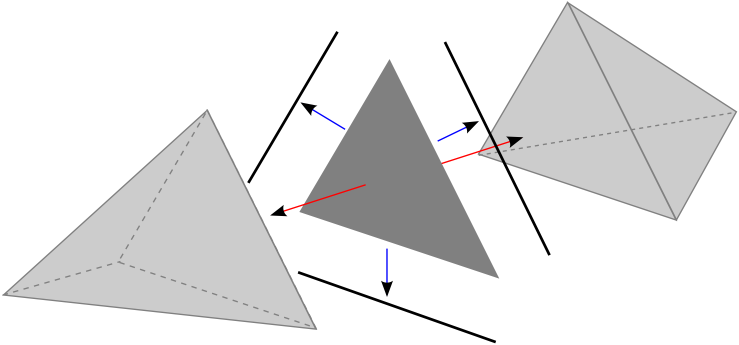
which implies:

$$\sigma^p = \left[\mathrm{vert}\left(\sigma^p\right)/\mathrm{vert}\left(\sigma_1^{p-1}\right), \ldots, \mathrm{vert}\left(\sigma^p\right)/\mathrm{vert}\left(\sigma_{p+1}^{p-1}\right)\right],$$

where:

$$\mathrm{vert}\left(\sigma^d\right) = \bigcup_{i=1}^{d+1} \mathrm{vert}\left(\sigma_i^{d-1}\right),$$

Our implemetation



It can be seen that:

- $C^*_{2,3}(\sigma^2) = C_{2,3}(\sigma^2)$,
- if σ^p $(p < 2)$ is 3-manifold, then $|C^*_{p,p+1}(\sigma^p)| = 1$.

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- **orient faces consistently/oppositely** – enforcing a consistent/opposite orientation on all $(p-1)$-faces of a p-simplex σ^p;

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- **star** – evaluation of the star of a simplex;
- **closure** – evaluation of the closure of a simplex or a set of simplices;
- **link** – evaluation of the link of a simplex;
- **boundary** – evaluation of the boundary of the simplex;
- **orient faces consistently/oppositely** – enforcing a consistent/opposite orientation on all $(p-1)$-faces of a p-simplex σ^p;
- **orient co-faces consistently/oppositely** – enforcing a consistent/opposite orientation on all $(p+1)$-simplices having a given p-simplex σ^p as a face;

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References

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