The non-central Nakagami distribution

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Abstract

The Nakagami distribution describe the square root of a random variable drawn from a Gamma distribution. Equivalently, the Gamma distribution can be seen as a one-dimensional Wishart distribution. In this note, we consider the distribution of the square root of a random variable drawn from a non-central one-dimensional Wishart distribution. We present the probability density function of this distribution along with closed-form expressions for its moments.

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1 The Nakagami distribution

Let $z_d \sim \mathcal{N}(0, \sigma^2), d = 1, \ldots, D$ be iid samples from a zero-mean normal distribution. Then

$$x = \sum_{d=1}^{D} z_d^2 \in \mathbb{R}^+ \tag{1.1}$$

follows a Gamma distribution. Equivalently, and more suitable for our purposes, x also follows a one-dimensional Wishart distribution [1],

$$x \sim \mathcal{W}_1(D, \sigma^2) \tag{1.2}$$

$$p(x) = \mathcal{W}_1(x \mid D, \sigma^2) = \sqrt{\frac{x^{D-2}}{2^D\, \sigma^{2D}}}\Gamma\left({}^{D}\!/\!{}_{2}\right)^{-1} \exp\left(-\frac{x}{2\sigma^2}\right). \tag{1.3}$$

This distribution can be seen as describing the squared norm of a zero-mean normally distributed vector. The norm of this vector is then given by $y = \sqrt{x}$, which follows a Nakagami distribution [2],

$$y \sim \text{Nakagami}\left({}^{D}\!/\!{}_{2}, D\sigma^2\right) \tag{1.4}$$

$$p(y) = \text{Nakagami}\left(y \mid {}^{D}\!/\!{}_{2}, D\sigma^2\right) = \frac{2}{\Gamma\left({}^{D}\!/\!{}_{2}\right)(\sqrt{2}\sigma)^D} y^{D-1} \exp\left(-\frac{y^2}{2\sigma^2}\right). \tag{1.5}$$

This expression is easily derived by the change of value theorem; see next section.

The expectation and variance of a Nakagami distributed variable is

$$\mathbb{E}\left[y\right] = \frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)}\sqrt{2}\sigma \tag{1.6}$$

$$\mathsf{var}\left[y\right] = \sigma^2\left(D - 2\left(\frac{\Gamma\left({}^{(D+1)}\!/\!{}_{2}\right)}{\Gamma\left({}^{D}\!/\!{}_{2}\right)}\right)^2\right). \tag{1.7}$$

We will derive a more general version of these results in the next section.

Preprint. Work in progress.

2 The non-central Nakagami distribution

Now, let $z_d \sim \mathcal{N}(\mu_d, \sigma^2), d = 1, \ldots, D$ be independent samples from normal distributions with variance σ^2. Then

$$x = \sum_{d=1}^{D} z_d^2 \in \mathbb{R}^+ \tag{2.1}$$

follows a one-dimensional non-central Wishart distribution [1],

$$x \sim \mathcal{W}_1(D, \sigma^2, \Omega) \tag{2.2}$$

$$\Omega = \frac{\sum_{d=1}^{D} \mu_d^2}{\sigma^2} \tag{2.3}$$

$$\begin{aligned} p(x) &= \mathcal{W}_1(x \mid D, \sigma^2, \Omega) \\ &= \sqrt{\frac{x^{D-2}}{2^D\, \sigma^{2D}}}\Gamma\left(D/2\right)^{-1} {}_0F_1\left(D/2, 1/4\Omega\sigma^{-2}x\right) \exp\left(-\frac{x}{2\sigma^2}\right) \exp\left(-\frac{\Omega}{2}\right). \end{aligned} \tag{2.4}$$

Here ${}_0F_1$ is a generalized hypergeometric function; see Sec. 7.3 of Muirhead's book [1]. This distribution can be seen as describing the squared norm of a normal distributed vector with isotropic variance. The norm of this vector is then given by $y = \sqrt{x}$, which we say follows a non-central Nakagami distribution. This distribution can be derived by change-of-variables to be

$$p\,(y) = \mathcal{W}_1(y^2 \mid D, \sigma^2, \Omega) \cdot 2y \tag{2.5}$$

$$= \frac{y^{D-1}}{2^{(D-2)/2}\, \sigma^D\, \Gamma\left(D/2\right)} {}_0F_1\left(\frac{D}{2}, \Omega\frac{y^2}{4\sigma^2}\right) \exp\left(-\frac{y^2}{2\sigma^2}\right) \exp\left(-\frac{\Omega}{2}\right). \tag{2.6}$$

To compute the moments of $\sqrt{x}$ we recall the following result from Muirhead [1, Theorem 10.3.7]. Let $\mathbf{X} \in \mathbb{R}^{q\times q}$ and $\mathbf{X} \sim \mathcal{W}_q(D, \mathbf{\Sigma}, \mathbf{\Omega})$ then

$$\mathbb{E}\left[(\det(\mathbf{X}))^k\right] = (\det(\mathbf{\Sigma}))^k 2^{qk} \frac{\Gamma_q(D/2 + k)}{\Gamma_q(D/2)} {}_1F_1(-k, D/2, -1/2\mathbf{\Omega}), \tag{2.7}$$

where ${}_1F_1$ is a generalized hypergeometric function. Since $y = \sqrt{x}$ is a positive scalar, its determinant is merely y, and we get

$$\mathbb{E}\left[\sqrt{x}^{2k}\right] = \sigma^{2k} 2^k \frac{\Gamma(D/2 + k)}{\Gamma(D/2)} {}_1F_1(-k, D/2, -1/2\mathbf{\Omega}). \tag{2.8}$$

From this we see that the mean and the variance of $y = \sqrt{x}$ is

$$\mathbb{E}\left[\sqrt{x}\right] = \sigma 2^{1/2} \frac{\Gamma((D+1)/2)}{\Gamma(D/2)} {}_1F_1(-1/2, D/2, -1/2\mathbf{\Omega}) \tag{2.9}$$

$$\mathbb{E}\left[x\right] = \sigma^2 2 \frac{\Gamma((D+2)/2)}{\Gamma(D/2)} {}_1F_1(-1, D/2, -1/2\mathbf{\Omega}) \tag{2.10}$$

$$\mathsf{var}\left[\sqrt{x}\right] = \mathbb{E}\left[x\right] - \mathbb{E}\left[\sqrt{x}\right]^2 \tag{2.11}$$

$$\begin{aligned} &= \frac{\sigma^2 2}{\Gamma(D/2)} \left(\Gamma\left(\frac{D+2}{2}\right) {}_1F_1(-1, D/2, -1/2\mathbf{\Omega}) \right. \\ &\qquad \left. - \frac{\Gamma((D+1)/2)^2}{\Gamma(D/2)} {}_1F_1(-1/2, D/2, -1/2\mathbf{\Omega})^2\right). \end{aligned} \tag{2.12}$$

Acknowledgments. The author was supported by a research grant (15334) from VILLUM FONDEN. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement nº 757360).

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