Exercises

12.1. **Quadratic Approximation for Poisson Data**
In Example 12.6 we stated (12.21) as a quadratic approximation of the ML estimation problem (12.20). Verify this approximation by using a second-order Taylor expansion of D_i at b_i given by

$$D_i(\tau) \approx D_i(b_i) + D_i'(b_i)(\tau - b_i) + \frac{1}{2} D_i''(b_i)(\tau - b_i)^2 , \qquad (12.42)$$

where D_i is defined as

$$D_i(\tau) = \exp(-b_i)\,\tau + \exp(-\tau) , \qquad i = 1, \ldots, m , \qquad (12.43)$$

and D_i' and D_i'' denote the first- and second-order derivatives of D_i, respectively.

12.2. **Tikhonov Solutions**
Define the objective function g for the Tikhonov regularization problem (12.3) by

$$g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2 + \alpha\frac{1}{2}\|\boldsymbol{x}\|_2^2. \qquad (12.44)$$

The gradient ∇g and the Hessian matrix $\nabla^2 g$ are defined in Eqs. (13.5) and (13.10), respectively.

1. Compute the gradient ∇g, and then show that $\nabla g = 0$ if and only if the normal equations (12.5) hold. Here you will need the relations

$$\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^T\boldsymbol{x},$$
$$\nabla(\boldsymbol{x}^T\mathbf{B}\boldsymbol{x}) = \mathbf{B}\boldsymbol{x} + \mathbf{B}^T\boldsymbol{x}$$

for a vector $\boldsymbol{x}$ and a matrix $\mathbf{B}$.

2. Compute the Hessian matrix $\nabla^2 g$, and then show that it is symmetric positive definite.

12.3. **Influence of Regularization Parameters on the Tikhonov Solutions**
In this exercise we use a small example to study the influence of the regularization parameter on the Tikhonov solutions. The matrix, the unperturbed right-hand side, and the unperturbed solution are

$$\boldsymbol{A} = \begin{pmatrix} 0.41 & 1.00 \\ -0.15 & 0.06 \end{pmatrix}, \quad \overline{\boldsymbol{b}} = \begin{pmatrix} 1.41 \\ -0.09 \end{pmatrix}, \quad \overline{\boldsymbol{x}} = \begin{pmatrix} 1.00 \\ 1.00 \end{pmatrix}.$$

Generate 25 random perturbations $\boldsymbol{b} = \overline{\boldsymbol{b}} + \boldsymbol{e}$ with the perturbation scaled such that $\|\boldsymbol{e}\|_2/\|\overline{\boldsymbol{b}}\|_2 = 0.15$. In MATLAB this computation takes the following form:

```
A = [0.41, 1.00; -0.15, 0.06];
x = [1.00; 1.00];
nfb = [1.41; -0.09];
norm_nfb = norm(nfb);

noise = randn(2,25);
norm_noise = sqrt(noise(1,:).^2 + noise(2,:).^2);
noise = noise./norm_noise*0.15*norm_nfb;
b = repmat(nfb,1,25) + noise;
```

Each column in b corresponds to one generated random perturbation. Then for each perturbed problem we computed the Tikhonov solutions defined in (12.6) with $\alpha = 0, 0.05, 0.5$, and 2.5. With a given α, we can use the following MATLAB commands to compute an array psol whose columns are the Tikhonov solutions for each column in b:

```
sm = A'*A+alpha*eye(2,2);
psol = sm\(A'*b);
```

Further, we calculate the Tikhonov solution without perturbations:

```
usol = sm\(A'*nfb);
```

Now let us plot the solutions:

```
figure,
plot(usol(1),usol(2),'r+', psol(1,:), psol(2,:),'b.')
axis([-0.5,2.5,0.2,1.6])
```

Observe how the sensitivity of the Tikhonov solutions to the perturbations changes when the regularization parameter α increases. What is your conclusion?

12.4. **Tikhonov Solutions in General Form**

Define the objective function g for the Tikhonov regularization problem in general form (12.27) by

$$g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2 + \alpha\frac{1}{2}\|\boldsymbol{D}\boldsymbol{x}\|_2^2 \,. \tag{12.45}$$

1. Compute the gradient ∇g, and derive the normal equation $\nabla g = 0$.
2. Compute the Hessian matrix $\nabla^2 g$, and then show that it is symmetric positive definite if the condition (12.33) holds.

12.5. **Finite Difference Approximation of the Gradient**

1. Consider a 1D function $x(t)$ on $0 \leqslant t \leqslant n$. Let $h = 1$ and $t_i = (i - 1/2)h$ for $i = 1, \ldots, n$. We discretize the function x as a vector $\boldsymbol{x} \in \mathbb{R}^n$ with

$x_i = x(t_i)$. Then the first-order derivative of x can be approximated by the forward finite difference scheme

$$x'(t_i) \approx \frac{x_{i+1} - x_i}{h}, \qquad i = 1, \ldots, n\,. \tag{12.46}$$

Assume a symmetric boundary condition, i.e., $x_{n+1} = x_n$. Using (12.46), show that a vector with values of the gradient x' at $t_1, \ldots, t_n$ can be approximated by $\boldsymbol{D}_n\,\boldsymbol{x}$, where $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ and $\boldsymbol{D}_n$ is defined as in (12.28).

2. Consider a 2D function $x(s,t)$ on $0 \leqslant s \leqslant M$ and $0 \leqslant t \leqslant N$. Let $h = 1$, $s_i = (i-1/2)h$, and $t_j = (j-1/2)h$ for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. We discretize the function x as a matrix $\mathbf{X} \in \mathbb{R}^{M\times N}$ with $x_{i,j} = x(s_i, t_j)$. Then the first-order partial derivatives along the vertical and horizontal directions can be approximated by the forward finite difference scheme

$$\frac{\partial x}{\partial s}(s_i, t_j) \approx \frac{x_{i+1,j} - x_{i,j}}{h}, \tag{12.47}$$

$$\frac{\partial x}{\partial t}(s_i, t_j) \approx \frac{x_{i,j+1} - x_{i,j}}{h} \tag{12.48}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. Assume a symmetric boundary condition, i.e., $x_{M+1,j} = x_{M,j}$ and $x_{i,N+1} = x_{i,N}$ for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. If we concatenate all columns in $\mathbf{X}$ to obtain a vector $\boldsymbol{x} \in \mathbb{R}^n$ with $n = MN$, show that the gradient $\nabla x = (\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t})$ can be approximated by $\boldsymbol{D}_{M\times N}\boldsymbol{x}$ with $\boldsymbol{D}_{M\times N}$ defined as in (12.29). Note that the first n entries in $\boldsymbol{D}_{M\times N}$ approximate $\frac{\partial x}{\partial s}$, and the last n entries approximate $\frac{\partial x}{\partial t}$.

12.6. **Importance of the Choice of Regularization Term**
This exercise is inspired by Figure 8.1 in [71]. Consider a simple ill-posed 1D inverse problem with missing data, where $\overline{\boldsymbol{x}} \in \mathbb{R}^P$ consists of samples of the sine function and the right-hand side $\overline{\boldsymbol{b}}$ is a subset of these samples:

$$\overline{\boldsymbol{b}} = \boldsymbol{A}\overline{\boldsymbol{x}}\,, \qquad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}_{\text{left}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}_{\text{right}} \end{pmatrix},$$

where $\boldsymbol{I}_{\text{left}}$ and $\boldsymbol{I}_{\text{right}}$ are two identity matrices. For example, in MATLAB you can use the following code:

```
P = 24;
t = linspace(0.2,2.5,P);
x = sin(t)';
l = p/3;
A = [eye(l),zeros(l,l),zeros(l,l);zeros(l,l),zeros(l,l),eye(l)];
b = A*x;
```

Based on the normal equations given in (12.5) and derived in Exercise 12.4, we can calculate the Tikhonov solution, as well as Tikhonov solutions in general form with $\boldsymbol{D} = \boldsymbol{D}_P$ defined as in (12.28). An alternative is to use $\boldsymbol{D} = \boldsymbol{D}_P^{(2)}$ with

$$\boldsymbol{D}_P^{(2)} = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{P \times P}, \tag{12.49}$$

which represents the second-order derivative with a symmetric boundary condition. In this test problem, we can simply set the regularization parameter $\alpha = 0.002$. Plot the reconstruction results and compare how the missing data is filled in when you use the three different regularization terms $\frac{1}{2}\|\boldsymbol{x}\|_2^2$, $\frac{1}{2}\|\boldsymbol{D}_P\boldsymbol{x}\|_2^2$, and $\frac{1}{2}\|\boldsymbol{D}_P^{(2)}\boldsymbol{x}\|_2^2$. Comment on the appearance of the reconstructions according to the corresponding regularization terms.

12.7. **TV for a 2D Function**
Consider a function $f(\mathbf{t})$ with the polar representation

$$f(r,\theta) = \begin{cases} 1, & 0 \leqslant r < R\,, \\ 1 + \frac{R}{h} - \frac{r}{h}, & R \leqslant r \leqslant R+h\,, \\ 0, & R+h < r. \end{cases}$$

This function is one inside the disk with radius $r = R$ and zero outside the disk with radius $r = R + h$, and it has a linear radial slope between zero and one. Verify the following expressions for the smoothing norms associated with the 1- and 2-norms of the gradient ∇f, which correspond to the anisotropic TV regularization and Tikhonov regularization in general form with the first-order derivative operator, respectively:

$$\left\|\nabla f\right\|_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\left|\frac{\partial f}{\partial t_1}\right| + \left|\frac{\partial f}{\partial t_2}\right|\right) \mathrm{d}t_1\,\mathrm{d}t_2 = 2\pi R + \pi h,$$

$$\left\|\nabla f\right\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial f}{\partial t_1}\right)^2 + \left(\frac{\partial f}{\partial t_2}\right)^2 \mathrm{d}t_1\,\mathrm{d}t_2 = \frac{2\pi R}{h} + \pi.$$

Referring to Example 12.13, if we use the 1-norm or 2-norm of ∇f as a regularizer, which kind of solutions should we expect?

12.8. **Numerical Computation of TV**
In this exercise, we implement the anisotropic and isotropic TV according to their definitions in (12.39) and (12.40), respectively. We first generate two test images of size 512×512 in MATLAB:

```
m = 512;
n = m;
Xtest = ones(m,n);
[X,Y] = meshgrid(linspace(-1,1,m),linspace(-1,1,n));

mask1 = (X(:).^2+Y(:).^2<=0.6);
mask1 = reshape(mask1, m, n);
Xtest1 = Xtest.* mask1;

mask2 = (abs(X(:))<=0.5) & (abs(Y(:))<=0.5);
mask2 = reshape(mask2, m, n);
Xtest2 = Xtest.*mask2;
```

By calling the MATLAB function `imagesc`, we can see that one test image is a disk and the other is a square.

Then we create two submatrices in the definition of $\boldsymbol{D}_{M\times N}$ in (12.29):

```
Dm = sparse(1:m-1, 1:m-1, -1, m, m)+sparse(1:m-1,2:m,1,m,m);
Dn = sparse(1:n-1, 1:n-1, -1, n, n)+sparse(1:n-1,2:n,1,n,n);

Dmn_1 = kron(speye(n),Dm);
Dmn_2 = kron(Dn,speye(m));
```

Now you are ready to finish the MATLAB codes for calculating the anisotropic and isotropic TVs according to their definitions in (12.39) and (12.40), respectively.